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Foundations of Classical Physics

Open Educational Resources

The Higher Education Academy Physical Sciences Centre

Welcome to this course in the Foundations of Classical Physics. It is made available as part of the Open Educational Resources Project (OER) run byt the HEA UK Physical Sciences Centre[1]

The course has been developed at the University of Edinburgh[2] in the School of Physics and Astronomy[3] over the past decade or so. It contains material from a variety of staff who have contributed to the development of the course, principally Alastair Bruce and Simon Bates[4] (who is the current course organiser for the equivalent first year undergraduate course at Edinburgh).

[1] http://www.heacademy.ac.uk/physsci/
[2] http://www.ed.ac.uk
[3] http://www.ph.ed.ac.uk
[4] http://www.ph.ed.ac.uk/~spb01
[5] http://creativecommons.org/licenses/by-nc-sa/2.5/scotland/

Lecture Notes

S0 The Tools of the Trade

Motivation: This introductory section explores what Physics is and reviews the key tools (mental, not metal) needed in the practice of Physics.

Objectives: By the end of this section you should

- understand the concepts of order of magnitude and significant figures;
- know the rules governing the use of units
- have consolidated and extended your knowledge of vectors
- have assimilated the strategies needed in tackling problems in physics

Key Mathematics: the use of vectors

S0.1 The trade: what is Physics?

[A] The aims of physics:

- to allow us to understand:
  - the world of our senses
  - the world of the very small
  - the world of the very large
  - the Big Picture embracing all length scales
- to allow us to do:
  - to go
  - to build
  - to heal

[B] The elements of physics:

- making observations ... which may be casual or systematic
- making models ... mathematical caricatures
- making sense ... linking observations with models through mathematics, using pen and paper or computer simulation

Learning Resources

- **Course Questions:** Thinking exercises.

S0.2 Units

[A] Units and standards

- Physics restricts itself to what is measurable in principle.
- Measurement is comparison with established unit. ⟵ EXAMPLES
- Units are interdependent. ⟵ EXAMPLE
- Units are defined through practical standards.
- Interdependence of units limits number of necessary standards. ⟵ EXAMPLE
- Accepted system of units and standards: Systeme Internationale
- Failure to abide by the rules can be very expensive[1]
- Precision of standards evolves with and for science. ⟵ I

[B] Rules

> KEY POINT **0.1**
> When two or more physical quantities are combined, the units combine in the same way.

⟵ EXAMPLE

> KEY POINT **0.2**
> Two physical quantities that are equated, added or subtracted must have the same units.

⟵ EXAMPLE

⟵ EXAMPLE

⟵ T

Learning Resources

- **Textbook:** HRW Chapter 1
- **Self-Test Questions:** available on–line

[1] http://www.seds.org/~spider/spider/Mars/ms98mco.html

S0.3 Numbers

[A] Order of magnitude

- The order of magnitude of a physical quantity is an estimate to within a power of 10.
- It is useful in assessing whether an effect is important or measurable.

←EXAMPLE

←X

[B] Precision

- The precision claimed for the value of some quantity is expressed in the number of digits ('significant figures', 'sf') used when it is quoted. ←EXAMPLE
- You must not write down 'insignificant' figures. ←EXAMPLE
- Experimental results are usually quoted with an error bound. ←EXAMPLE

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Learning Resources

- **Self-Test Questions:** available on–line
- **Course Questions:** Making estimates.

S0.4 Vectors

[A] Definitions

A vector is a quantity which has both magnitude (positive) and direction.
A scalar is a quantity which has magnitude (positive or negative) only.

[B] Notation

Vector status is shown variously by arrows ($\vec{A}$) or bold font (**A**).
The magnitude of the vector $\vec{A}$ is denoted by $|\vec{A}|$ or simply A . ←EXAMPLES

←I

[C] Utility

Vectors allow the laws of physics to be formulated concisely. ←I

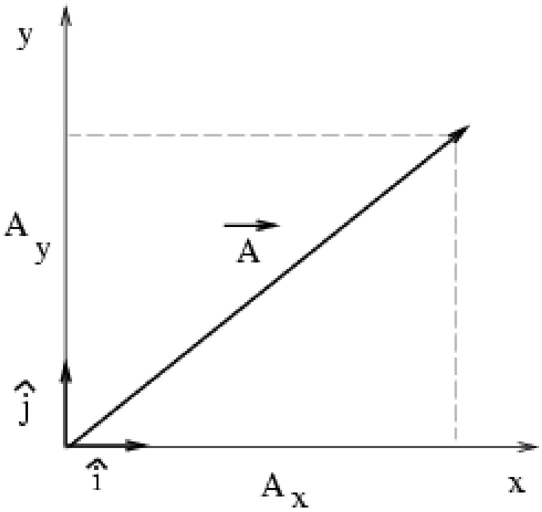
[D] The vector sum

- The sum of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} + \vec{B}$
- The magnitude and direction of $\vec{C}$ are defined geometrically by the 'nose-to-tail' construction.
- Distinguish carefully between $\vec{A} + \vec{B}$ and $A + B$.

← EXAMPLE
← M

[E] Components and unit vectors

- A vector can be represented by a set of components referred to a coordinate system.
- Consider a (two-dimensional: 2D) rectangular coordinate system, defined by two mutually perpendicular (x, y) axes.
- Any vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$



where

- A_x and A_y are the x and y components of $\vec{A}$ (the projections of $\vec{A}$ on the x and y axes)
- $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors (vectors of unit magnitude) along x and y axes.

- Use of the component-representation allows vector addition 'by algebra'.

← A
← EXAMPLE

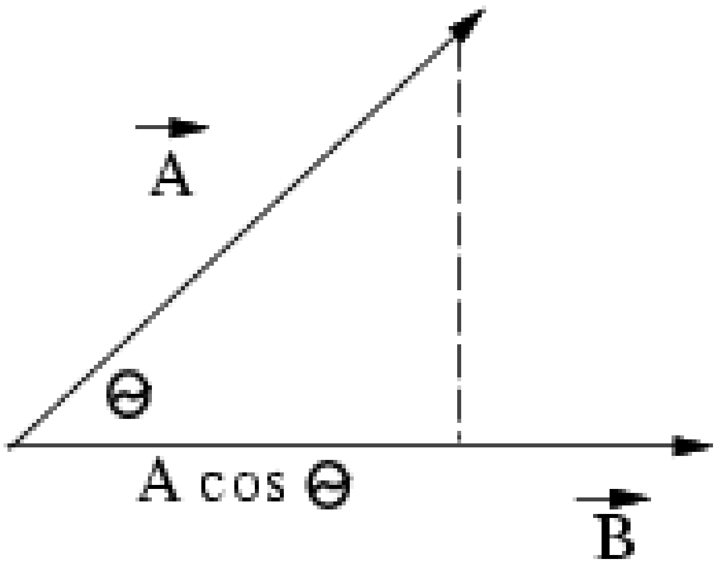
[F] The dot product

KEY POINT **0.3**
The **dot product** (or **scalar product**) of two vectors $\vec{A}$ and $\vec{B}$:

- is written as $\vec{A} \cdot \vec{B}$
- is a **scalar**
- is given by $\vec{A} \cdot \vec{B} = AB\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

- **Visualization**

  The dot product of two vectors reflects the size of the projection ('shadow') of one on the other.



← EXAMPLE

- Dot products of unit vectors:
  $\hat{\imath} \cdot \hat{\jmath} = 1 \times 1 \times \cos(90) = 0$
  $\hat{\imath} \cdot \hat{\imath} = 1 \times 1 \times \cos(0) = 1$

- Hence the component representation (in 2D):

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath}) = A_xB_x + A_yB_y$$

[G] Cross product

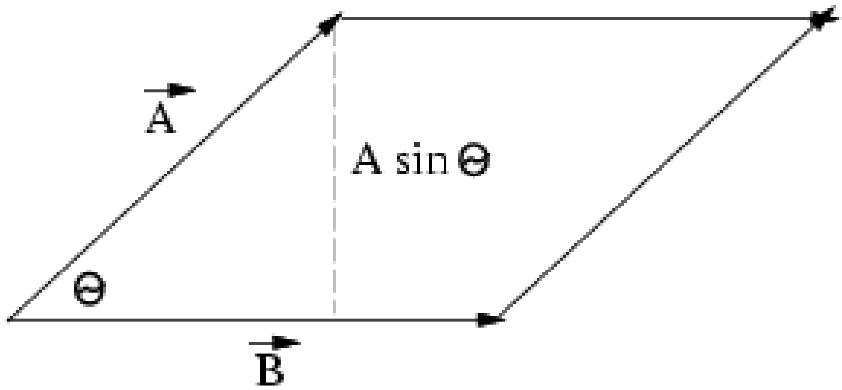
KEY POINT **0.4**
The **cross product** (or **vector product**) of $\vec{A}$ with $\vec{B}$:

- is written as $\vec{A} \times \vec{B}$
- is a **vector**
- has the magnitude $AB \sin\theta$ where θ is the (smaller) angle between $\vec{A}$ and $\vec{B}$
- has direction perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in the sense in which a corkscrew would move if turned so as to take $\vec{A}$ into $\vec{B}$ (the corkscrew rule).

← A

- **Visualization**

The cross product of two vectors is (in magnitude) the area of the parallelogram with sides formed from the two vectors.



← EXAMPLE

← MORE?

[H] One-dimensional vectors

> KEY POINT **0.5**
> A one-dimensional (1D) vector is represented by a scalar whose sign (positive/negative) indicates the direction (right/left) along the 1D axis.

← EXAMPLE

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Learning Resources

- **Textbook:** HRW Chapter 3
- **Self-Test Questions:** available on–line
- **Course Questions:** Thinking about vectors, Adding vectors, Multiplying vectors: the dot (or scalar) product, Unit vectors and dot products, Multiplying vectors: the cross (or vector) product.

S0.5 Problem solving

In this course we will invite you to solve many physics problems. We do so for two reasons. First, the experience will provide you with an opportunity to apply the principles of physics, and so to understand more fully what they mean. Second, the experience will also help you learn more about the art of problem solving itself –the stuff of active science.

Practising the art is the best way of learning it: but a frequent plea when confronted with a problem is – we have heard it many times – 'I don't know where to start'. This is particularly relevant when the problems become more complex than those you might have encountered thus far in your Physics.

It's a little like learning to drive a HGV (... I am assuming that none of you can...) When first behind the wheel, it's sort of similar to a car, but much different as well. More levers, more gears, harder to go round corners..... What will help you gain confidence (an important skill in Physics as well as HGV driving...!) is to have a **strategy** that will help you get moving. With practice, you'll be enacting the elements of the strategy automatically. Unfamiliar roads will not faze you and your rig will rumble along wherever you choose to take it.

This section presents a formal problem solving strategy that will allow you approach solving a problem systematically. We'll identify a number of general guidelines that are helpful in keeping you on the right track. They are important for all the problem-solving you will do in this course, and beyond. So they are set out fully. After we have set them out and discussed them in general terms we shall illustrate them in action in the context of a specific worked example.

[A] The strategy - F-D-P-E-E

The strategy comprises five distinct steps......

- **F**ocus on the problem (understand the problem)
- **D**escribe the physics (analyse the problem)
- **P**lan a solution (work out a strategy)
- **E**xecute the plan
- **E**valuate the result

Let's look at each one in a bit more detail, along with some guidelines that give you a bit more detail about what to actually 'do'

[B] Focus on the problem

GUIDELINE **0.1 Draw a sketch**
We can often appreciate more clearly what is involved in a problem by reexpressing it in pictorial form. Most problems in mechanics cry out for a picture, since they are concerned with phenomena that we can easily imagine 'seeing'. However even in the most abstract realms, such as quantum physics, and relativity, physicists draw pictures (symbolic, schematic) to act as problem-solving props.

X

GUIDELINE **0.2 Choose a sensible notation**
In most problems you will find that you need to choose symbols to respresent the important physical quantities. You will need to do this even if those quantities are given specific numerical values in the question: you will see why in Guideline 0.3. Symbols need to be explicitly defined, either by indicating their role in your sketch or in words. Choosing symbols that are simple and evocative makes life easier. Thus 'm' and 'M' are wonderful as masses but would be lousy as lengths!

[C] Model the problem

GUIDELINE **0.3 Identify the principles**
Most of the problems you will meet (in this course) involve the application of just one or perhaps two basic principles. Your task is to identify which. Sometimes this will be straightforward: perhaps the problem is 'like' one you have seen before; perhaps there are key phrases that point you in the right direction. Sometimes you will find it harder, as you come to deal with problems that are less idealised.

GUIDELINE **0.4 Formulate the equations**
While we can go a long way with words and pictures, the full power of our physical principles is only released when we express them in the language of mathematics. Solving problems usually means translating them into a mathematical form, and invoking the tools of mathematics to follow through the consequences.

[D] Plan your strategy

GUIDELINE **0.5 Devise a strategy**
Once you've written down the basic equations expressing what you know, you need to take time to <u>plan</u> how you will manipulate them to find what you actually want to know. Time spent on this kind of route-planning may spare you the pain of finding yourself on unmade roads, at dead ends or the wrong destination. Do you have all the information that you need to be able to solve the problem?

GUIDELINE **0.6 Do not substitute numbers until you must**
Knowing the units of the terms in an equation provides an important check on whether the equation <u>can</u> be right: you lose this check as soon as you substitute numbers. You also lose the information about special cases: see Guideline 0.10

← A TIP!

[E] Execute the strategy

GUIDELINE **0.6 Use quantities in consistent units**
In numberical problems, make sure than any quantities you are combining have the same units. As an example, in a kinematics problem, using a speed of miles per hour to calculate the time taken for some event will yield an answer in hours! Maybe seconds would be a better choice. In some problems, there will be no numbers specified - in this case, solve the algebra, don't 'invent' numbers for quantities.

[F] Evaluate the answer or result

GUIDELINE **0.8 Ask if the answer makes sense**
This is perhaps the single most important rule of the lot! It means different things in different circumstances. Sometimes it means asking whether a numerical result is physically reasonable. It usually involves other more specific tests, which we will deal with separately...

GUIDELINE **0.9 Check the units**
Our second rule governing the use of units (KP 0.2) provides an invaluable check which you can put into practice throughout any bit of algebraic manipulation —and always at the end. Remember that a valid equation must pass this test: if it fails, there is something wrong and it is not sensible to go further until you have sorted it.

GUIDELINE **0.10 Appeal to special cases**
Quite frequently you will find that you know some other condition that your answer has to satisfy, when you think of some special case (or limit, or range) of the physical parameters... another reason for carrying through the argument in terms of the symbols, instead of the specific values they may have in the particular instance. While satisfying this special case condition doesn't guarantee that you're correct, it makes it that much more likely.

GUIDELINE **0.11 Be prepared to look from a different angle.**
There are some problems which are tough –perhaps even impossible– to do if you broach them in what appears to be the 'obvious' way; but which become quite simple when you find a different way of looking at them. The most challenging and satisfying problems often need this kind of flexibility of thought.

You'll get plenty of practice enacting this strategy in the weekly workshops. For now, here's a worked example to get started..... ⟵ EXAMPLE

⟵ 1

Learning Resources

- **Textbook:** HRW provides a series of 'Problem solving tactics', spread through the text.

  In addition to this, we'll be providing some dissections of (real) good and bad answers in video format online. (These will be linked from the homepage when available).
- **Self-Test Questions:** available on–line
- **Course Questions:** Asking yourself if it makes sense.

S1 Space and Time

Motivation: Physics deals with the sequence of events that make up the unfolding story of the universe. The most basic questions we can ask about 'events' are 'where?' and 'when'. Thus Space and Time are the key concepts of physics. In this section we explore the classical view of Space and Time developed by Galileo and Newton, and touch on its failures, unearthed by Einstein.

Objectives: By the end of this section you should

- be familiar with the key concepts of kinematics, and their geometrical significance
- be competent in the use of vectors in describing 1D and 2D motion
- know and be able to apply the equations describing motion at constant acceleration and motion in a circle at constant speed
- know the Galilean description of relative motion and be aware of its limitations

Key Mathematics: derivatives and slopes; integrals and areas

S1.1 One dimensional particle kinematics

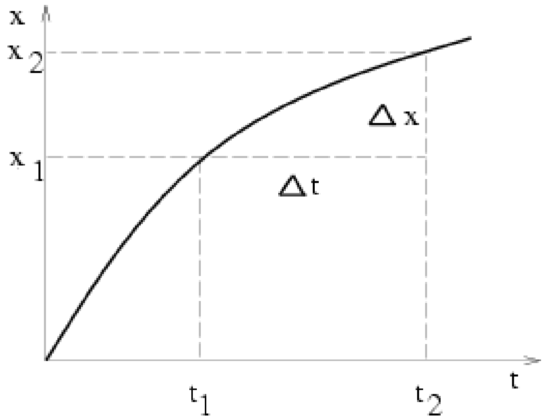
[A] Context

- We focus on the kinematics of a particle moving in one dimension ('1D', or 'd=1'). ← COMMENTARY
- Fundamental concern: variation of position (x) with time (t) ← COMMENTARY

← I

[B] Displacement

Consider a particle whose x coordinate varies smoothly but arbitrarily with t.



- Suppose particle moves from position x_1 at time t_1 to x_2 at t_2.
- We define the time interval

$$\Delta t = t_2 - t_1 \tag{1.1}$$

- We define the associated displacement by

$$\Delta x = x_2 - x_1 \tag{1.2}$$

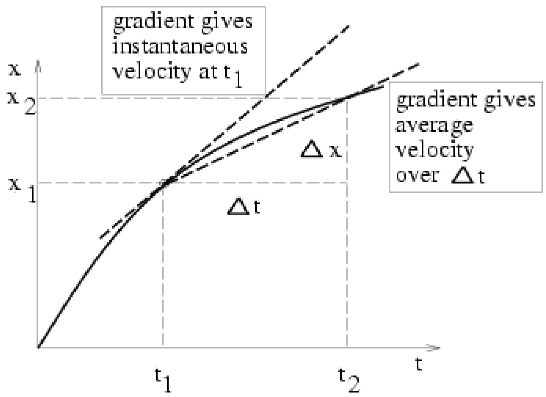
← COMMENTARY

← EXAMPLE

[C] Velocity

From the variation of x with t we define the average velocity over a time interval as

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} \tag{1.3}$$



KEY POINT **1.1**
The instantaneous velocity is defined as the average velocity over the next infinitesimally small time interval:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

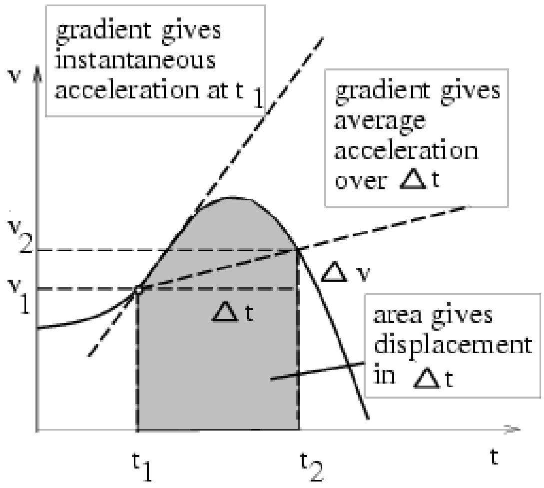
Geometrical significance: v is gradient of x-t graph.

← COMMENTARY

[D] Acceleration

Now consider the variation of v with t. The average acceleration over a time interval is defined as

$$a_{\text{av}} = \frac{\Delta v}{\Delta t} \qquad (1.4)$$



KEY POINT **1.2**
The instantaneous acceleration is the average acceleration over the next infinitesimally small time interval:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Geometrical significance: a is gradient of v-t graph.

← COMMENTARY

← EXAMPLE

[E] Integral forms of key equations

KEY POINT **1.3**
The integral forms of the $x - t$ and $v - t$ relationships are:

$$\Delta x = \int_{t_1}^{t_2} v dt \quad \text{and} \quad \Delta v = \int_{t_1}^{t_2} a dt$$

In particular the displacement is the area under the $v - t$ curve.

← ANALYSIS

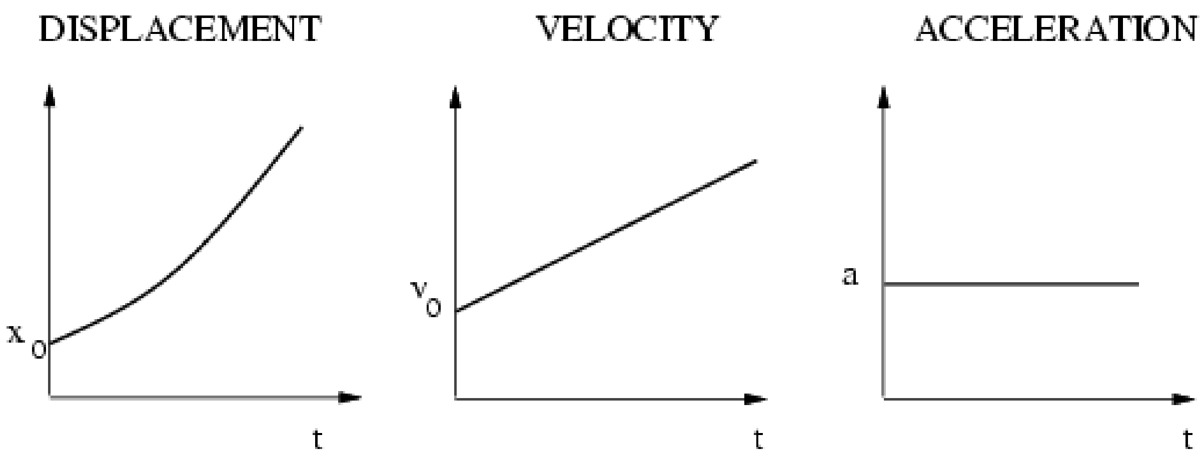
[F] Constant acceleration equations

KEY POINT **1.4**
For 1D motion at constant acceleration a, the position and velocity (x and v) at the end of a time interval (t) are related to those at the beginning of the interval (x_0 and v_0) by

$$\begin{aligned} v &= v_0 + at \qquad (a) \\ x - x_0 &= v_0 t + \frac{1}{2}at^2 \qquad (b) \\ v^2 &= v_0^2 + 2a(x - x_0) \qquad (c) \end{aligned}$$

← ANALYSIS
← M

Visualization



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Learning Resources

- **Textbook:** HRW Chapter 2
- **Self-Test Questions:** available on–line
- **Course Questions:** The meaning of derivatives and integrals, Vertical motion under gravity.

S1.2 Kinematics in two (or three) dimensions

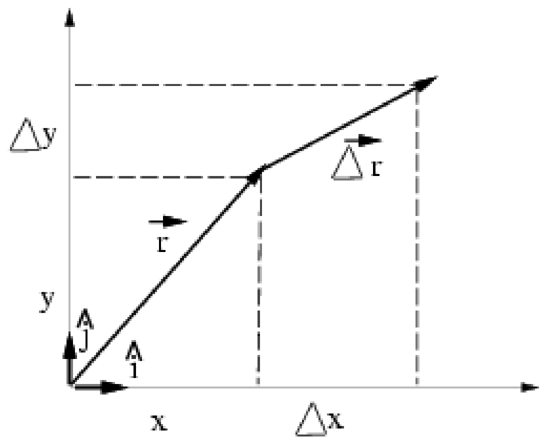
[A] Position and displacement vectors

- In space dimension d = 2 (or 3...) the position of a particle is specified by:
  - d coordinates (x, y ...)

or

- a d-dimensional vector $\vec{r}$

Visualization



The position vector can be written as

$$\vec{r} = x\hat{i} + y\hat{j}$$

The displacement vector is

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

[B] The velocity vector

KEY POINT **1.5**
The velocity is the time rate of change of the position vector; it is a vector:

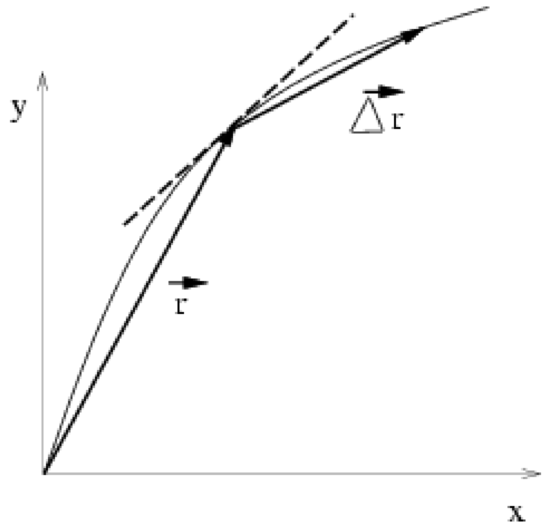
$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \text{In component form:} \quad v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt}$$

COMMENTARY

- **Visualization**

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

  - As $\Delta t \to 0$, $\Delta r \to 0$
  - $\Delta \vec{r}$ becomes tangent
  - And so ...



KEY POINT **1.6**
The velocity vector is always tangential to the particle path.

← A

[C] The acceleration vector

KEY POINT **1.7**
The acceleration is the time rate of change of the velocity; it is a vector:

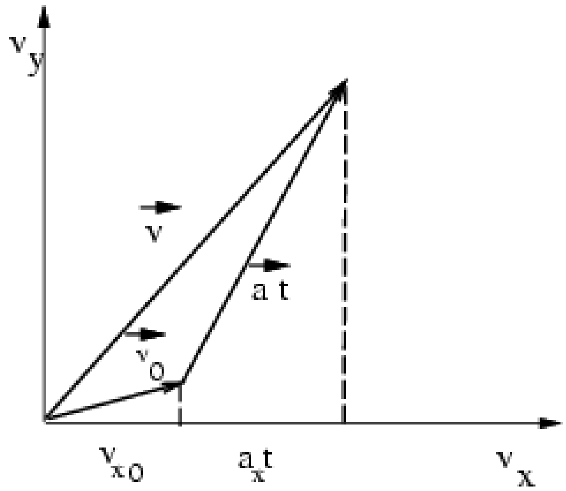
$$\vec{a} = \frac{d\vec{v}}{dt} \qquad \text{In component form:} \quad a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt}$$

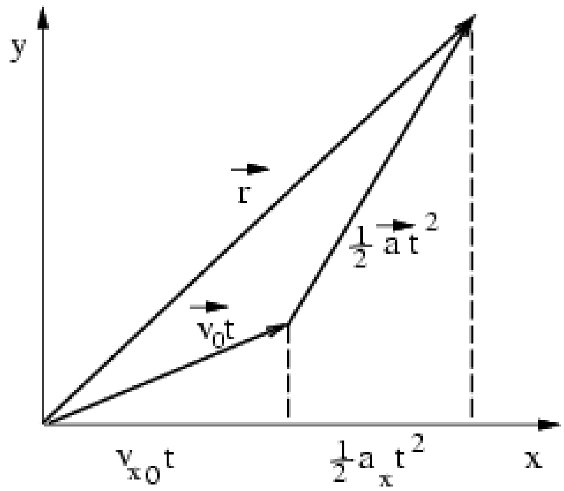
- **The acceleration is non-zero if**
  - the velocity vector is changing in **direction**
  - the velocity vector is changing in **magnitude**

[D] Constant acceleration equations

- If the acceleration vector is constant, the 1D kinematic equations (KP 1.4) can be applied to the motion associated with **each** of the axes. ← EXAMPLE

- **Visualization**





Learning Resources

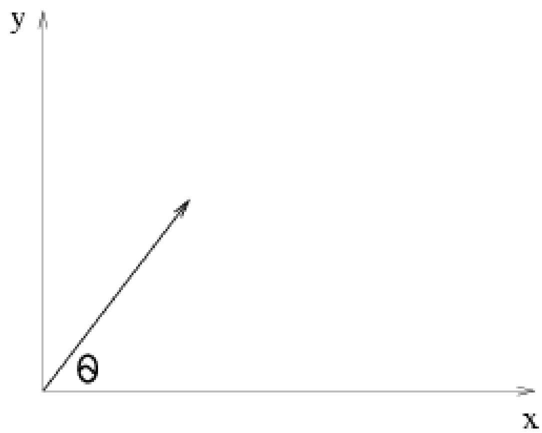
- **Textbook:** HRW Chapter 4.1-4
- **Self-Test Questions:** available on–line

S1.3 Application: projectile motion

[A] The problem

Consider motion of projectile near the earth's surface.

- Choose coordinate axes:
  x - horizontal
  y - vertical



- Specify initial conditions:
  - launch at $t = 0$ from $x = 0$, $y = 0$
  - launch velocity: $\vec{v}_0$

- components:

$$v_{x0} = v_0 \cos\theta \quad \text{and} \quad v_{y0} = v_0 \sin\theta \tag{1.5}$$

←M

- Identify assumptions:
  - in words: acceleration is down, magnitude g
  - in equations: $a_x = 0 \qquad a_y = -g$

[B] Results

$$x\text{-motion:} \qquad v_x = v_{x0} \quad \text{and} \quad x = v_{x0}t \tag{1.6}$$

$$y\text{-motion:} \qquad v_y = v_{y0} - gt \quad \text{and} \quad y = v_{y0}t - \frac{1}{2}gt^2 \tag{1.7}$$

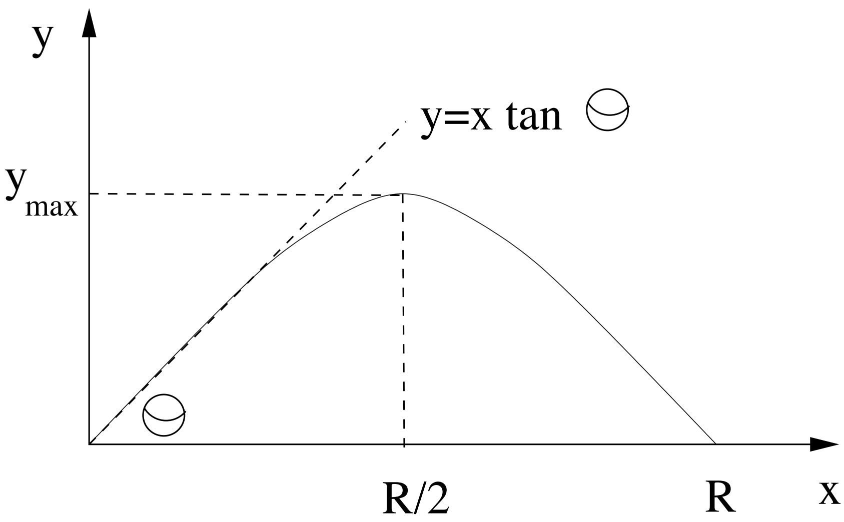
$$\text{time of flight:} \qquad t_f = \frac{2v_0 \sin\theta}{g} \tag{1.8}$$

$$\text{range:} \qquad R = \frac{2v_0^2 \sin\theta\cos\theta}{g} \tag{1.9}$$

$$\text{trajectory equation:} \qquad y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta} \tag{1.10}$$

←ANALYSIS

Visualization:



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Learning Resources

- **Textbook:** HRW Chapter 4.5-6
- **Self-Test Questions:** available on–line
- **Course Questions:** The trajectory equation, Projectile motion, Steep and shallow trajectories.

S1.4 Application: circular motion

[A] Uniform circular motion

> KEY POINT **1.8**
> A particle moving in a circle of radius r at uniform speed v has an acceleration of magnitude v^2/r, directed towards the centre of the circle (centripetal).

← COMMENTARY

← ANALYSIS

[B] Generalisations

- **If the path is not circular**: the result holds at each point on the path, with r the radius of curvature at that point. ← COMMENTARY

- **If the speed is not constant**: there is also a component of acceleration of magnitude $\frac{dv}{dt}$ tangential to the path.

← COMMENTARY

← T

← A

Learning Resources

- **Textbook:** HRW Chapter 4.7
- **Self-Test Questions:** available on–line
- **Course Questions:** A consolidation exercise, Centripetal acceleration, Average acceleration, Trajectory curvature, Centripetal and tangential acceleration.

S1.5 Relativity: the common sense view

[A] Context

- The description given to the motion of any object depends upon the perspective of the observer. ← EXAMPLE

- A **reference frame** is the name given to the coordinate system to which an observer (or group of observers) refers measurements. ← EXAMPLES

← I

- **Relativity** is concerned with the relationship between measurements referred to different reference frames.

[B] Results: Galilean transformations

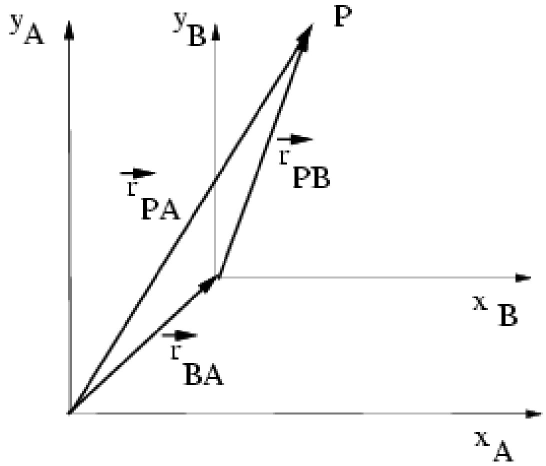
KEY POINT **1.9**
The relationships between the positions, velocities and accelerations of a particle, P, assigned in two reference frames, A and B, in uniform relative motion are

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

$$\vec{a}_{PA} = \vec{a}_{PB}$$

These are the Galilean transformations.
They are based on the assumption that time is simple.



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[C] Status of results

These results are

- consistent with common sense
- practically correct for ‘slow’ kinematics
- wrong for ‘fast’ kinematics, where ‘fast’ signifies involvement of speeds comparable with speed of light, c.

T

Learning Resources

- **Textbook:** HRW Chapter 4.8-9
- **Self-Test Questions:** available on–line
- **Course Questions:** Relative velocities 1, Relative velocities 2, Choosing a speed limit.

S1.6 Relativity: Einstein's view

This section is included for general interest. It is not part of the examinable programme of the course.

Learning Resources

- **Textbook:** HRW Chapter 37 takes you through the first steps in Special Relativity.

S2 Force Mass and Motion

Motivation: Understanding a changing world means understanding motion. This section is concerned with the key concepts (mass, force) underlying the classical Newtonian theory of motion, and expressed in Newton's three laws. We illustrate the application of these laws in the context of a wide range of forces, and touch on some of the curious 'forces' encountered in 'accelerating' reference frames.

Objectives: By the end of this section you should

- understand the roles played by force, mass and inertial reference frames in the laws of motion
- be familiar with a wide variety of the forces encountered in nature
- be able to apply Newton's laws to analyse the behaviour of systems experiencing such forces
- understand, qualitatively, the fictitious forces experienced in non-inertial reference frames

Key Mathematics: vectors; a glimpse of differential equations

S2.1 Inertial reference frames: Newton's 1st Law

←PREAMBLE

KEY POINT **2.1**
Newton's 1st Law. A body free of external 'influences' will have constant velocity (zero acceleration) with respect to any inertial reference frame.

- This statement defines the term inertial reference frame
  – an inertial reference frame is one in which the 1st Law holds!
- A remote star is an ideal inertial frame
  – because it is itself 'free of external influences'. ←Q

- Any reference frame moving at constant velocity w.r.t. an inertial frame is also inertial – the Galilean transformations (KP 1.9) guarantee this. ← HELP?

- The earth is approximately an inertial frame. ← COMMENTARY ← I ← T

Learning Resources

- **Textbook:** HRW Chapter 5.1-3

S2.2 Force and mass: Newton's 2nd and 3rd laws

[A] Preamble

- We introduce the concept of force to quantify external influence on acceleration.

- We introduce the concept of mass to quantify inertia (resistance to acceleration).

- 2nd and 3rd laws together:
  - allow us to define force and mass
  - provide framework for explaining motion under given forces.

[B] The 2nd and 3rd laws

KEY POINT **2.2**
Newton's 2nd Law: The acceleration $\vec{a}$ a body displays (with respect to an inertial reference frame) is related to the net external force $\vec{F}$ it experiences by

$$\vec{F} = m\vec{a}$$

where m is its (inertial) mass. This is a vector equation.

KEY POINT **2.3**
Newton's 3rd Law: The force $\vec{F}_{12}$ exerted on body 1 by body 2 is equal in magnitude but opposite in direction to the force $\vec{F}_{21}$ exerted on body 2 by body 1:

$$\vec{F}_{12} = -\vec{F}_{21}$$

The two forces constitute an 'action-reaction pair'. They act on different bodies.

← COMMENTARY

← THE SCALE OF FORCES

Learning Resources

- **Textbook:** HRW Chapter 5.4-6, 5.8
- **Self-Test Questions:** available on–line

S2.3 How to use Newton's Laws

This section gathers together some hints you will find useful when applying Newton's Laws. They complement the general problem-solving guidelines set out in S0.5 . They are best learned by working through the examples which follow.

- **Choose your system**: Newton's 2nd Law can be applied to any 'bit' of the universe. You must choose the 'bit': this is your system. It may be all or only part of the physical system described in the problem. It is a good idea to draw a box around it to remind you what you have chosen.

- **Identify all the forces**: exerted on the chosen system by anything else. Other forces are irrelevant to the behaviour of the system you have chosen. ←A TIP?

- **Draw a vector diagram**: showing all the forces you have identified; each force should be represented by a vector with its origin on the system. This is called a free-body diagram.

- **Apply the 2nd Law to your chosen system**: to establish the relationship between the net force it experiences and its acceleration. Then:

  - If the acceleration is given, the task is to infer something about the forces.
  - If the forces are given the task is to find and solve the equation for the acceleration. This is the equation of motion. Its solution describes the way the position coordinates of the system change with time.

- **Remember that forces are vectors**: ...and have to be combined accordingly. In doing so you can make life easier for yourself by choosing your coordinate axes wisely. ←I

- **Remember that the 3rd law is different**: in that it always involves two systems. One of the forces in the third law pair acts on one system; the other force acts on the other system. ←A TIP?

←T

Learning Resources

- **Textbook:** HRW sets out its own set of Problem Solving Tactics, in relation to Newton's Laws, in Chapter 5.9
- **Course Questions:** Forces are vectors, Forces and accelerations, Choosing the right system, Misapplying the Third Law, Actions and reactions.

S2.4 Classification of forces

- To make use of Newton's 2nd law, the equation
$$m\vec{a} = \vec{F}$$
specifying the acceleration, needs to be augmented by an equation
$$\vec{F} = \ldots$$
specifying the force.

- Nature's forces fall into three categories: fundamental, phenomenological and fictitious.

[A] Fundamental forces

- These are forces of interaction between elementary constituents of the universe:
  - gravitational force, between point masses
  - electrostatic force, between point charges
  - some others

←X

- They are fully specified by some quantitatively explicit force law.

- Forward reference: S2.11 and S2.12

←EXAMPLE

[B] Phenomenological forces

- These are forces of interaction between macroscopic portions of matter.

- They represent the aggregate effects of fundamental forces.

- They can generally be described only by equations containing empirically-determined constants.

- Forward reference: S2.8

←EXAMPLES

[C] Fictitious forces

- These are forces we invent to make sense of our experience in non-inertial reference frames.
- Forward reference: S2.13

Learning Resources

- **Textbook:** HRW Chapter 5.7, 6.5

S2.5 Gravitational force near the earth's surface

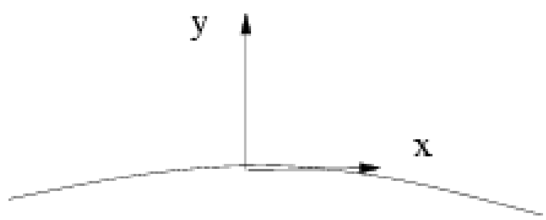
[A] About the force

The gravitational force on a body of mass m within a small enough region near the earth's surface is:

$$\vec{F}_G = -mg\hat{y}$$

where

- $\hat{y}$ is a unit vector in the direction of the 'local' vertical
- g is local gravitational acceleration

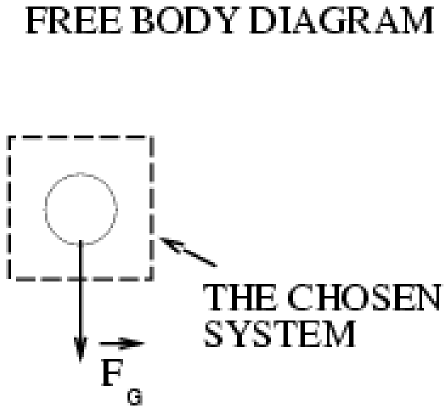


[B] Example problem

Determine the motion of a body of mass m experiencing only a uniform gravitational force, and consider the implications of the 3rd law.

Solution

- **Application of 2nd law**



Choose the system to be the body only

- the second law:

$$m\vec{a} = \vec{F}_G$$

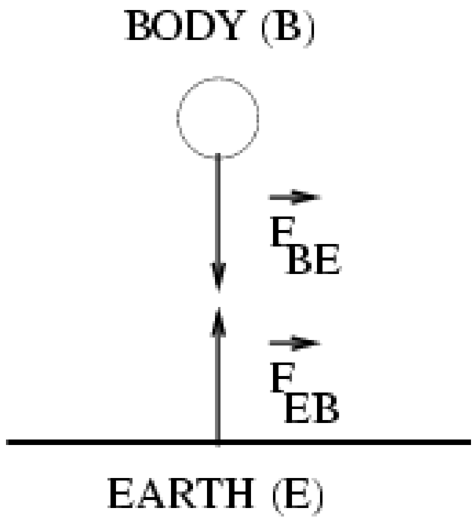
- the force law:

$$\vec{F}_G = -mg\hat{y}$$

- the equation of motion:

$$\vec{a} = -g\hat{y}$$

- The solution: already established in KP 1.4.
- **Application of 3rd law**



- choose the two systems: body, B and earth, E
- the third law:

$$\vec{F}_{BE} = -\vec{F}_{EB} = -mg\hat{y}$$

←A

[C] Inertial and gravitational mass

- In Newton's 2nd Law

$$\vec{F} = m\vec{a}$$

m is strictly the inertial mass.

- In the gravitational force equation

$$\vec{F}_G = -mg\hat{y}$$

m is strictly the gravitational mass.

- The claim that these masses are the same leads to the prediction that all bodies have the same free fall acceleration.

←X

- This claim is one expression of the Principle of Equivalence

- Forward reference: S2.13

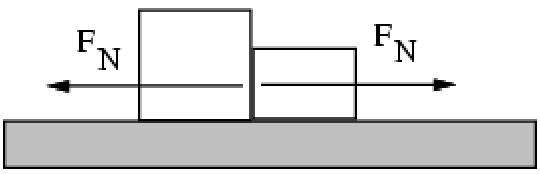
S2.6 Normal contact force

[A] About the force

The normal contact force F_N is the repulsive force of interaction between two surfaces in contact, acting at right angles to the surfaces, and inhibiting closer contact.

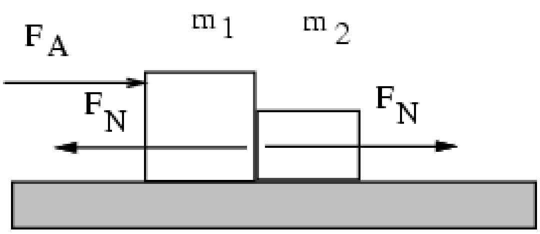
- otherwise known as: 'normal force', or 'normal reaction'

- each body experience force of same magnitude but opposite direction.



[B] Example problem

Two bodies are free to move on a smooth horizontal surface under the action of a horizontal force of magnitude F_A.



Determine the acceleration and the normal contact force between them.

←SOLUTION

Results

acceleration:

$$a = \frac{F_A}{m_1 + m_2}$$

magnitude of normal force

$$F_N = F_A \frac{m_2}{m_1 + m_2}$$

Check It!

- Are the units OK? ☐
- Does it make sense? ☐

← T
← T
← T
← M

Learning Resources

- **Textbook:** HRW Chapter 5.7,5.9
- **Self-Test Questions:** available on–line
- **Course Questions:** Normal contact force.

S2.7 Tension

[A] About the force

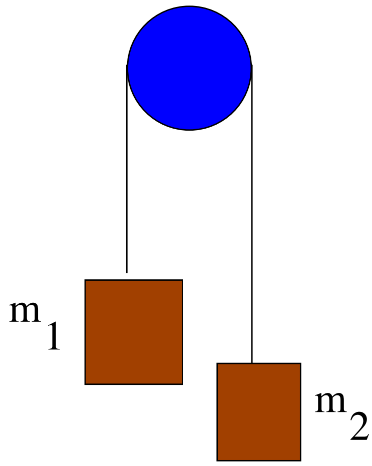
- A stretched string (or wire, rod . . .) is said to be under tension.
- The force such a string exerts on an object to which it is attached has magnitude T, the tension, and acts along the string, away from the object.
- If the string is light enough the tension is uniform (the same at both ends).

← COMMENTARY

[B] Example problem

Two masses m_1 and m_2 are suspended over a light and frictionless pulley. Find the accelerations of the masses and the string tension.

← SOLUTION



Results

string tension:

$$T = \frac{2m_1 m_2}{m_1 + m_2} g$$

downward acceleration of m_1 (and upward acceleration of m_2):

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Check It!

- Are the units OK? ☐
- Does it make sense? ☐

←T

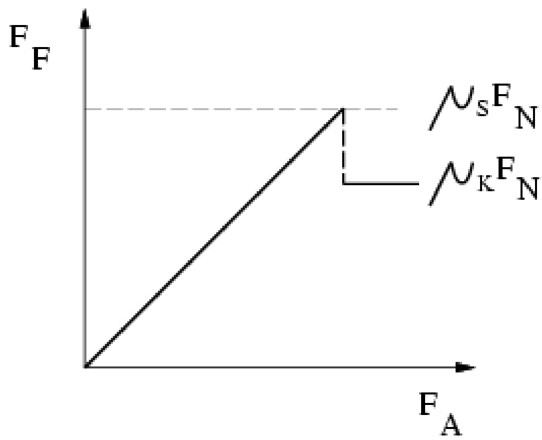
Learning Resources

- **Textbook:** HRW Chapter 5.7,5.9
- **Self-Test Questions:** available on–line
- **Course Questions:** Tension and normal forces, Tensions.

S2.8 Frictional force

[A] About the force

- The friction force F_F is the interaction force between two surfaces in contact, acting parallel to the surfaces, inhibiting sliding. ←I
- The value of F_F depends both on the applied force parallel to the surface, F_A and the normal contact force between the surfaces, F_N:
  - no sliding:
    $$F_F = F_A$$
  - on the point of sliding:
    $$F_F = \mu_s F_N = F_A$$
  - while sliding:
    $$F_F = \mu_k F_N < F_A$$

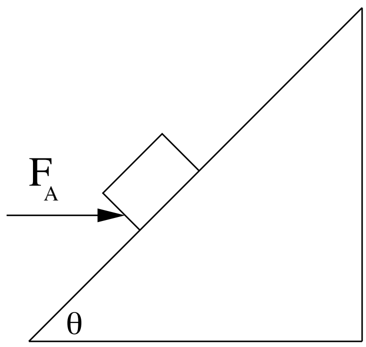


- The quantities μ_s and μ_k are the coefficients of static and kinetic friction.
- They depend on the surfaces in contact; μ_k is smaller than μ_s.

← COMMENTARY

[B] Example problem

A block of mass $m = 2\,kg$ rests on a plane inclined at an angle $\theta = 45^\circ$ to the horizontal. A force $\vec{F}_A$ of magnitude $30\,N$ is applied to it, horizontally. The block moves up the plane with constant velocity.



Write down equations for (a) the normal contact force between box and plane and (b) the net force acting up the plane. Deduce the value of the coefficient of kinetic friction. ← SOLUTION

Results

$$\mu_k = \frac{F_F}{F_N} = \frac{F_A\cos\theta - mg\sin\theta}{F_A\sin\theta + mg\cos\theta} = \frac{30-20}{30+20} = 0.2$$

Check It!

- Are the units OK? ☐
- Does it make sense? ☐

← T

← M

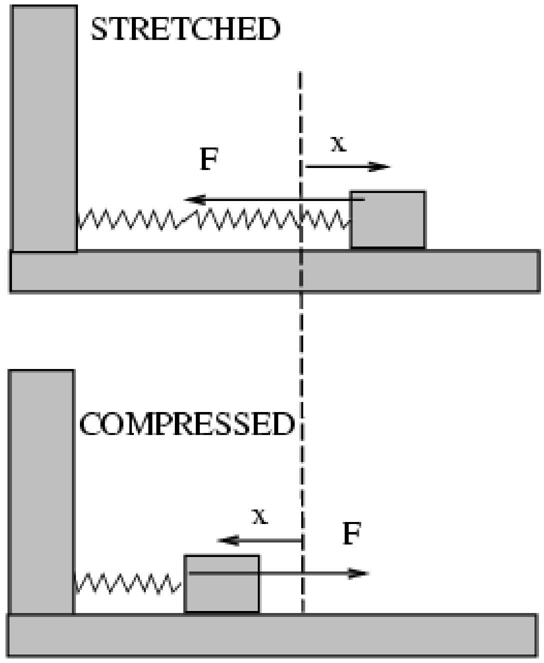
Learning Resources

- **Textbook:** HRW Chapter 6.2-6.3
- **Self-Test Questions:** available on–line
- **Course Questions:** Frictional forces.

S2.9 Linear restoring force

- A stretched or compressed spring exerts a force on a body to which it is attached.

- The magnitude of the force is proportional to the distortion of the spring x provided x is small (Hooke's Law).



- The force exerted by the spring on the object is

$$F = -kx$$

  - sign means force acts opposite to displacement: it is restoring
  - force depends on the first power of x: it is linear

- Equation of motion of such a body is

$$ma_x = m\frac{d^2x}{dt^2} = -kx$$

- This is the generic equation of simple harmonic motion (SHM).
- Forward reference: S6 .

Learning Resources

- **Self-Test Questions:** available on–line

S2.10 The centripetal force

[A] About the force

- A body of mass m moving at uniform speed v in a circle of radius r exhibits an acceleration of magnitude v^2/r towards the centre (KP 1.8).
- By Newton's 2nd law this acceleration must be attributed to a force F_C where:
  - the magnitude of F_C is $F_C = mv^2/r$

- the direction is towards the centre

- The generic name for this force is centripetal.

← EXAMPLES

[B] Example problem:

See S2.11 ← T

← M

Learning Resources

- **Textbook:** HRW Chapter 6.5
- **Self-Test Questions:** available on–line
- **Course Questions:** Banking on friction.

S2.11 The gravitational force

← PREAMBLE

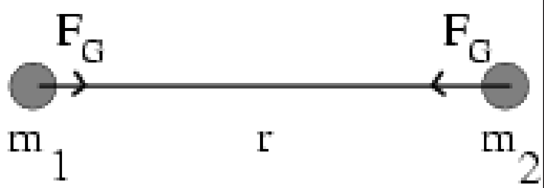
[A] About the force

KEY POINT **2.4**
The gravitational force of interaction between two point masses m_1 and m_2 separated by distance r:

- has magnitude

$$F_G = \frac{Gm_1 m_2}{r^2}$$

with G the universal constant of gravitation.



- is attractive

← I

[B] Example problem

A planet of mass m moves in a circular orbit of radius r about the sun, of mass M. Establish the relationship between the period of the orbit and its radius. Treat the sun and planet as point masses. ← I

← SOLUTION

Results

In words: the square of the period is proportional to the cube of the orbit radius.
This is Kepler's Law of Periods. ← I

Explicitly:

$$T^2 = \frac{4\pi^2}{GM} r^3$$

← COMMENTARY

Learning Resources

- **Textbook:** HRW Chapters 13.2 and 13.7 deal with Kepler's Laws, including the Law of Periods. They rest on concepts, notably angular momentum conservation, to be discussed in S5 .
- **Self-Test Questions:** available on–line
- **Course Questions:** G and g, Geostationary orbits.

S2.12 Electrostatic forces

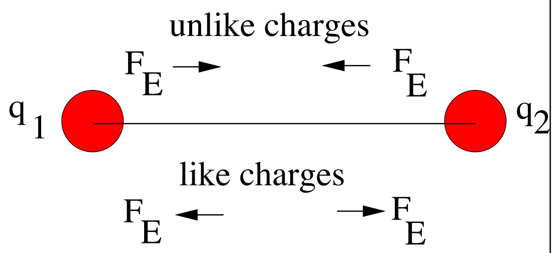
← PREAMBLE

[A] About the force

KEY POINT **2.5**
The electrostatic (Coulomb) force of interaction between two point charges q_1 and q_2 separated by distance r:

- has magnitude

$$F_E = \frac{K \mid q_1 q_2 \mid}{r^2}$$



with K a fundamental constant of electrostatics

- is attractive if q_1, q_2 are unlike
- is repulsive if q_1, q_2 are like

← COMMENTARY

[B] Example problem

Estimate the ratio of the gravitational force between two electrons and the electrostatic (Coulomb) force between two electrons.

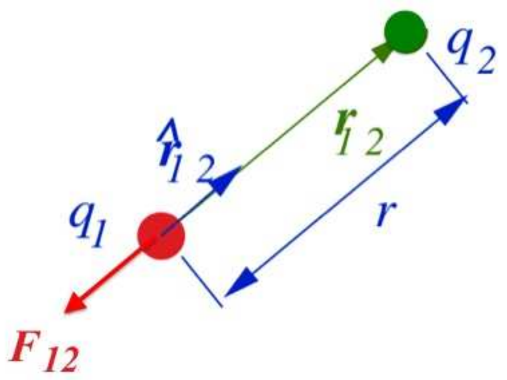
← SOLUTION

[C] Dealing with multiple charges - the principle of superposition

In the same way that we have been adding forces as vectors throughout this section, electrostatic forces are no exception and serve as an illustration of the Principle of

Superposition in action.

Let's start with the vector form of the force due on one charge from another. If we have two point charges q_1 and q_2 seperated by a vector $\vec{r}_{1\,2} = r\,\hat{r}_{1\,2}$ as shown below



3.1

then the force on q_1 as aresult of q_2 is a *vector* giving,

KEY POINT **2.6**
Coulomb's Law in vector form, being:

$$\vec{F}_{1\,2} = -\frac{1}{4\pi\,\epsilon_0}\frac{q_1\,q_2}{r^2}\,\hat{r}_{1\,2}$$

where $\hat{r}_{1\,2}$ is a unit vector from q_1 in the direction of q_2.

Note that this force is directed away from q_1 if both charges have the same sign and towards q_2 if opposite. Generalising to a series of charges, for example $q_1 \to q_6$

Then the force on q_1 as a result of the charges $q_2 \to q_6$ will be a **vector sum**, being

KEY POINT **2.7**

$$\vec{F}_1 = \vec{F}_{1\,2} + \vec{F}_{1\,3} + \cdots = \sum_{j=2}^{N} \vec{F}_{1\,j}$$

where each force $\vec{F}_{1\,j}$ is the force on q_1 as a result of the j^{th} charge being given by KP 2.6.

[D] The Electric Field

We have just seen how to calculate the force on a charged particle due to the presence of a second charge. But if the two charges are nowhere near each other how is that force 'felt' by the charge? How can there be this action at a distance?

We can explain this by saying that particle 2 sets up an **electric field** in the space all around it. Paticle 1 is affected by this field. Thus particle 2 exerts a force on particle 1 not by touching it but by the electric field its presence has created. ← SCALAR VS VECTOR FI

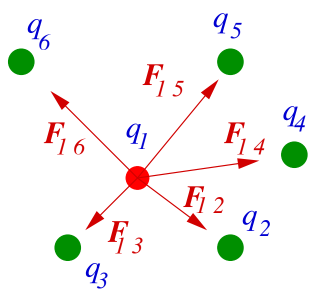
We can define the electric field $\vec{E}$ at a point, P, in the vicinity of a charge by considering the electrostatic force acting on a test positive charge of q_0 at P.

KEY POINT **2.8**

$$\vec{F} = q_0 \, \vec{E}$$

The **magnitude** of the field at this point is given by $e = \frac{F}{q_0}$ and the **direction** is that of the force that acts on the test positive charge.

If we then extend this idea to a collection of charges as we did previously for the electrostatic force



3.2

Then we can write the force on this charged particle as,

$$\vec{F}_1 = q_1 \, \vec{E}_T$$

where $\vec{E}_T$ is the total Electric Field resulting from the other charges. Since the force on the particle F_1 is a linear sum of contributions from all other particles, we can show that the *Electric Field*

KEY POINT **2.9**

$$\vec{E}_T = \vec{E}_2 + \vec{E}_3 + \cdots = \sum_{j=2}^{N} \vec{E}_j$$

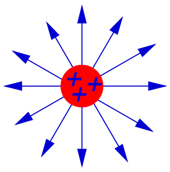
where $\vec{E}_j$ is the **Electric Field** due to the j^{th} charge, so being,

$$\vec{E}_j = \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_j^2} \hat{r}_{1\,j}$$

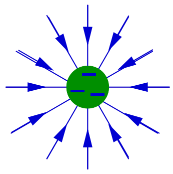
So the Electric Field seen by charge q_1 is a **Linear Vector Summation** of the Electric Fields due to the other charges; the principle of superposition again.

[E] Field lines

Field lines provide a convenient way for us to visualise the vector nature of an electric field. They are lines of force which show the direction of the field at a point in space, with the separation between the lines indicating the magnitude or strength.



(a) Positive Charge

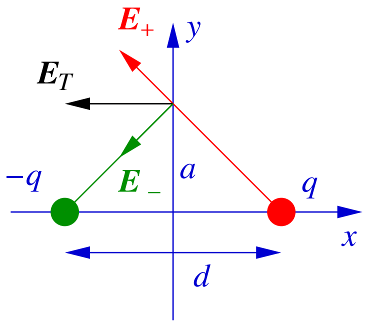


(b) Negative Charge

3.3

← ANALYSIS

Now consider two charges of $+q$ and $-q$ separated by a fixed distance d as shown. This configuration is known as an **Electric Dipole**.



3.4

← MATHEMATICAL ANAL

KEY POINT **2.10**
The total **Electric Field** is given by

$$\vec{E}_T = -\frac{1}{4\pi\epsilon_0}\frac{q\,d}{r^3}\,\hat{\imath}$$

which is parallel to the x-axis and pointing from the **positive** towards the **negative** charge, which is what you expect from the diagram.

Considering KP 2.10 we see that the **Electric Dipole** is characterised by the **vector** quantity

$$\vec{p} = q\,d\,\hat{\imath}$$

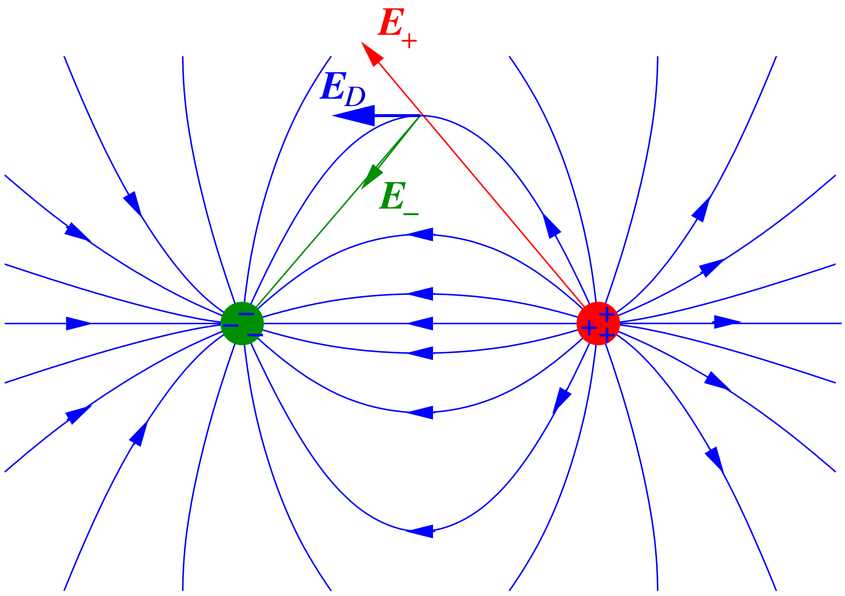
which is known as the **Electric Dipole Moment**, which has units of $\mathrm{C\,m}$.

Note the direction of $\vec{p}$, it is from the negative towards the positive end of the dipole. So in terms of the **Electric Dipole Moment** the Electric Field a distance a perpendicular to the dipole axis is given by,

KEY POINT **2.11**

$$\vec{E}_T = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{r^3}$$

We can repeat this calculation at all point in space and form the field lines from a dipole as shown



3.5

←A

Learning Resources

- **Textbook:** The treatment of electrostatics starts in HRW in Chapter 21, and much of the next 11 chapters is concerned with developing towards a treatment of Maxwell's Equations. However, we will start (and remain) in the foothills, covering Chapter 21 and selected bits of Chapter 22.
- **Self-Test Questions:** available on–line
- **Course Questions:** Balancing electrostatic forces , Inside the nucleus, Zero net field.

S2.13 Fictitious forces

← PREAMBLE

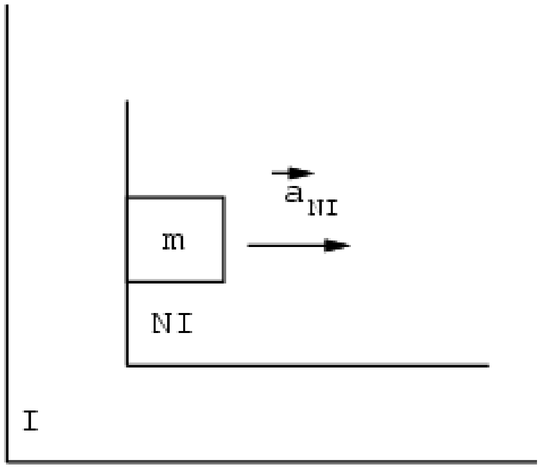
KEY POINT 2.12
Any reference frame accelerating w.r.t. an inertial frame is non-inertial.
When the motion of some body is described from the perspectives of a non-inertial reference frame Newton's laws hold only if we introduce fictitious forces which reflect the acceleration.
Such forces are indistinguishable from gravitational forces (The Principle of Equivalence).

← X

← ANALYSIS

[A] Body at rest in accelerating frame

- Consider body of mass m stationary in a non-inertial (NI) frame
- Suppose NI frame has acceleration $\vec{a}_{NI}$ with respect to inertial frame, I.



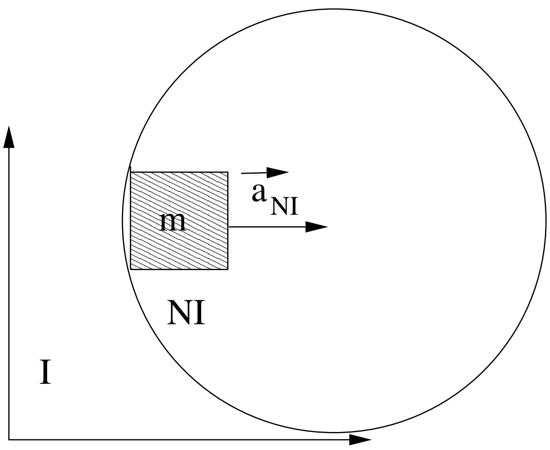
- Then behaviour of the body w.r.t. NI obey's Newton's Laws but with a fictitious force

$$\vec{F}_{NI} = -m\vec{a}_{NI} \tag{2.1}$$

← ANALYSIS

[B] Body at rest in a rotating frame

- This is a special case of [A] where $\vec{a}_{NI}$ is the centripetal acceleration.

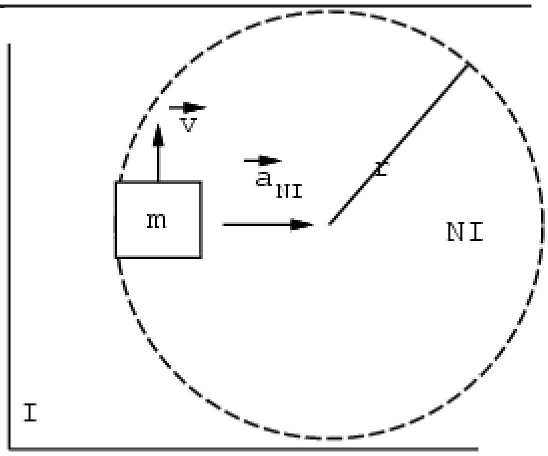


- Observers in rotating frame must invent a fictitious force which
  - has magnitude

$$F_{NI} = mv^2/r$$

  - acts out from centre of rotation

This is the centrifugal force.

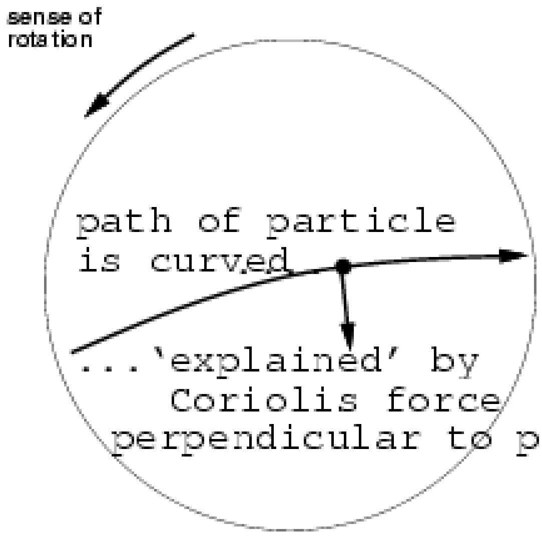


← WARNING
← I
← A

[C] Body moving in a rotating frame

In addition to the centrifugal force a body moving in a rotating frame 'experiences' a second fictitious force which:

- depends on its speed in the rotating frame;
- acts at right angles to its path;
- makes the path curved with respect to the rotating frame.



This is the Coriolis force.

← ANALYSIS
← I
← A

Learning Resources

- **Textbook:** Conspicuously absent from HRW. Those with a desire to find out more may want to consult Serway and Jewett (Physics, 6th ed) on pages 159-161
- **Self-Test Questions:** available on–line
- **Course Questions:** Perspectives, The Principle of Equivalence.

S3 Energy and Work

Motivation: To describe the changing world around us, we must describe its state. Energy is one of the key tools that allows us to do this. In this section we explore the concept of energy: its definition, its conservation and its utility in problem solving.

Objectives: By the end of this section you should

- know how work, kinetic energy and potential energy are defined
- understand why a potential energy can only be defined for a conservative force
- be able to use conservation of energy to solve simple problems

Key Mathematics: integration and differentiation

S3.1 Introduction

[A] What is energy?

- Energy is a scalar quantity associated with the state of a system; work is associated with changes in that state. The ideas of work and energy provide a new angle from which to view Newtonian mechanics, and a powerful set of tools for solving problems.

KEY POINT **3.1**
Energy is defined as the ability of a system to do work.

- Energy can come in two forms–

.... kinetic

KEY POINT **3.2**
Kinetic energy is the energy a body has by virtue of its motion

....and potential

KEY POINT **3.3**
Potential energy is the energy a system stores as a result of its state, shape or position.

← EXAMPLES

← DEMO

- The concepts of energy and energy conservation, although consistent with Newtonian mechanics, are valid in regimes and for systems where Newtonian mechanics does not hold; at those extreme speeds and tiny distances where relativity and quantum mechanics reign.

S3.2 Work

← PREAMBLE

[A] Definition

- When a force acts on a body to produce a displacement, work is done. Work is a scalar quantity measured in newton-meters (Nm) or Joules (J).

- If a constant force $\vec{F}$ acts on a body to produce a straight line displacement $\vec{d}$, the amount of work W done on the body by the force is given by the dot product (KP 0.3):

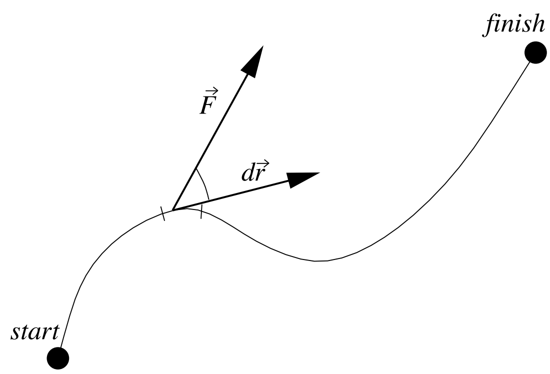
$$W = \vec{F} \cdot \vec{d} \tag{3.1}$$

- If the force $\vec{F}$ is not constant, and the resultant displacement is not a straight line, the amount of work that the force does is:

KEY POINT **3.4**
Work:

$$W = \int_{\text{start}}^{\text{finish}} \vec{F} \cdot d\vec{r}$$

• This is the sum, over the entire path, of all the infinitesimal bits of work done along all the infinitesimal displacements $d\vec{r}$, by the force $\vec{F}$ which is, in general, different at every point along the path.

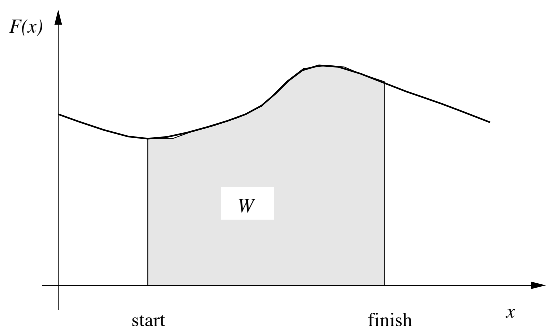


← NOTE

- In one dimension, with a varying force $F(x)$

$$W = \int_{\text{start}}^{\text{finish}} F(x)dx \tag{3.2}$$

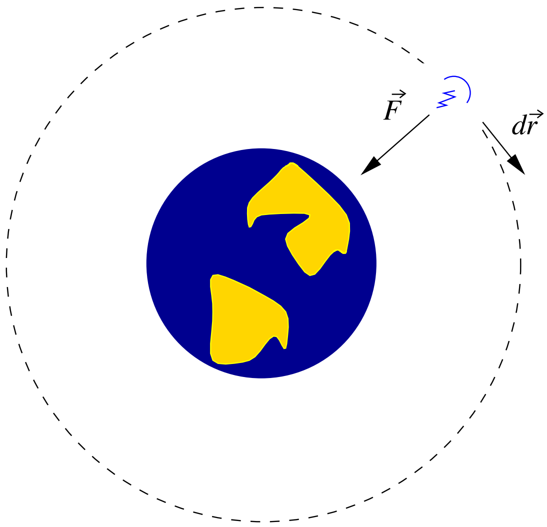
• The work done, W, by the applied force, $F(x)$, between the start and finish points is the area under the curve.



[B] Traps for the unwary!

- Work is defined as the work done by the force on the body.
- It can be positive or negative, depending on the sign of the force and the displacement. ←EXAMPLES
- A force acting perpendicular to the direction of motion of a body does no work, since the scalar product between $\vec{F}$ and $d\vec{r}$ is zero.

Example: the force of gravity on a satellite orbiting in a circle is always perpendicular to the displacement. Forces that do no work are common in circular motion (S2.10).



- 'Hard work' is somtimes nothing of the sort ←DEMO

Learning Resources

- **Textbook:** HRW Chapter 7. HRW starts with kinetic energy, then moves on to define work. Otherwise the treatments are identical.
- **Self-Test Questions:** available on–line
- **Course Questions:** Work, force and displacement I, Work, force and displacement II, Work as an integral.

S3.3 Power : the rate of working

←PREAMBLE

[A] Definition

- Power is defined as the rate at which work is done. If an amount of work W is done in time Δt, the average power over that interval is defined to be

KEY POINT **3.5**
Average Power:

$$P_{av} = \frac{W}{\Delta t}$$

- The instantaneous power is the instantaneous rate of doing work:

KEY POINT **3.6**
Instantaneous Power:

$$P = \frac{dW}{dt}$$

- The SI unit of power is the watt (W), which is one joule per second (Js^{-1}).

←COMMENTARY

←T

Learning Resources

- **Textbook:** HRW chapter 7.9
- **Self-Test Questions:** available on–line
- **Course Questions:** Power.

S3.4 Kinetic energy

[A] The work - energy theorem

- The work - energy theorem <u>defines</u> kinetic energy.

KEY POINT **3.7**
Work-energy theorem. When a mass m is accelerated by a force along some path, the total work done on the mass by the force is

$$W = \int_i^f \vec{F} \cdot d\vec{r} = K_f - K_i$$

where the initial and final kinetic energies are K_i and K_f respectively.

KEY POINT **3.8**
Kinetic energy is the work done to accelerate a particle from rest to velocity $\vec{v}$

$$K = \frac{1}{2}mv^2$$

←ANALYSIS

←EXAMPLE

Learning Resources

- **Textbook:** HRW 7.5
- **Self-Test Questions:** available on–line
- **Course Questions:** Work and kinetic energy I, Work and kinetic energy II, Variable force.

S3.5 Potential energy

[A] Introduction

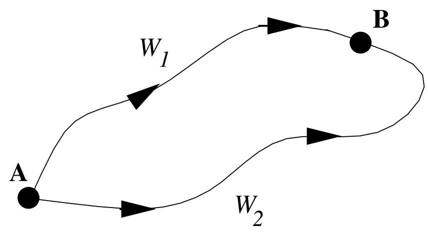
- Potential energy is the energy a system stores as a result of its state, shape or position (KP 3.3).
- A potential energy can only be defined for a conservative force.

← EXAMPLES

[B] Conservative forces

KEY POINT **3.9**
If the work done by a force in moving an object between two states is independent of the path taken between those two states, the force is conservative.

Necessarily, the total work done by a conservative force in moving an object around a closed path is zero.



For a conservative force, the work done in moving between **A** and **B** does not depend on the path

- Examples of conservative forces include:
  - the gravitational force (S2.5);
  - the electrostatic (Coulomb) force;
  - linear restoring ('spring') forces (S2.9).
- Frictional forces, such as static friction, dynamic friction (S2.8) and drag are not conservative: these forces dissipate energy (i.e. remove energy from the system). The work they do depends on the path followed by the force.

← I
← COMMENTARY
← WORKED EXAMPLE

[C] Potential energy

- Because the work done by a conservative force in moving a body between the two states is unique, we can assign a number to every state, (i.e. a function), that tells us the work done in moving from an arbitrarily chosen reference state to the current state.
- We call this function the potential energy. The work done in moving between any two states is then the difference in potential energy of the two states.

KEY POINT **3.10**
Potential Energy. The difference in potential energy, ΔU, between initial and final states is defined as the negative of the work done by the associated force

$$\Delta U = U_f - U_i = -\int_i^f \vec{F} \cdot d\vec{r}$$

- Note: The potential energy is often referred to as U rather than ΔU. However, there is always an implied reference state.

← COMMENTARY

[D] Forces from potential energies

- The force acting at any point can be determined from the potential energy.

KEY POINT **3.11**
Force from potential: In one dimension, force is related to potential energy via:

$$F = -\frac{dU}{dx}$$

- A convenient way of thinking about forces and potential energies is to imagine walking around a hilly landscape. ← D

Learning Resources

- **Textbook:** HRW Chapter 8.1 - 8.3. (HRW 8th Ed has a new section on Reading a P.E. curve, 8.6)
- **Self-Test Questions:** available on–line
- **Course Questions:** Conservative forces, Force and potential energy.

S3.6 Potential energy: examples

- In this course, you will meet a few different types of potential energy:

- energy stored in a spring; ← I
- gravitational potential energy; ← A
- energy stored in a wave; ← I

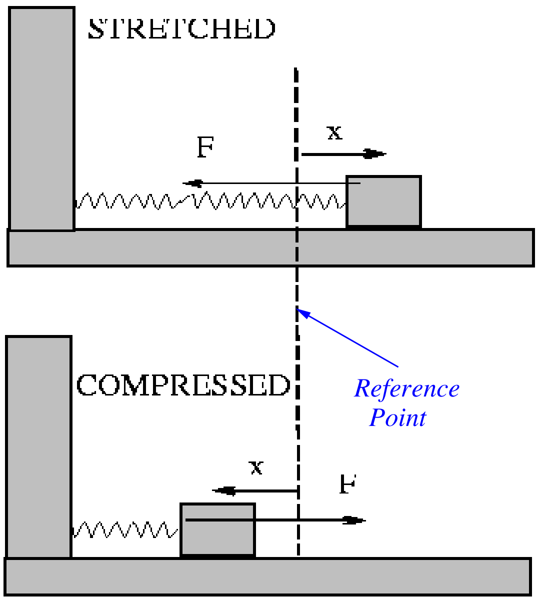
- There are many, many other sorts of potential energy:
  - energy stored in chemical bonds; ← EXAMPLE
  - electrical energy stored in a capacitor;
  - magnetic energy stored in a field;
  - energy stored in excited states of atoms;

[A] Linear (spring) force: potential energy

- The force a spring exerts on a body is proportional to the displacement of the spring (S2.9).

> KEY POINT **3.12**
> The potential energy stored in a spring with displacement x is
> $$U = \frac{1}{2}kx^2$$

Note: convention dictates we choose the reference point to be the displacement from the equilibrium length.



← ANALYSIS
← WORKED EXAMPLE

[B] Gravitational potential energy of a body near the earth's surface

- Near the earth's surface the gravitational force is constant (S2.5).

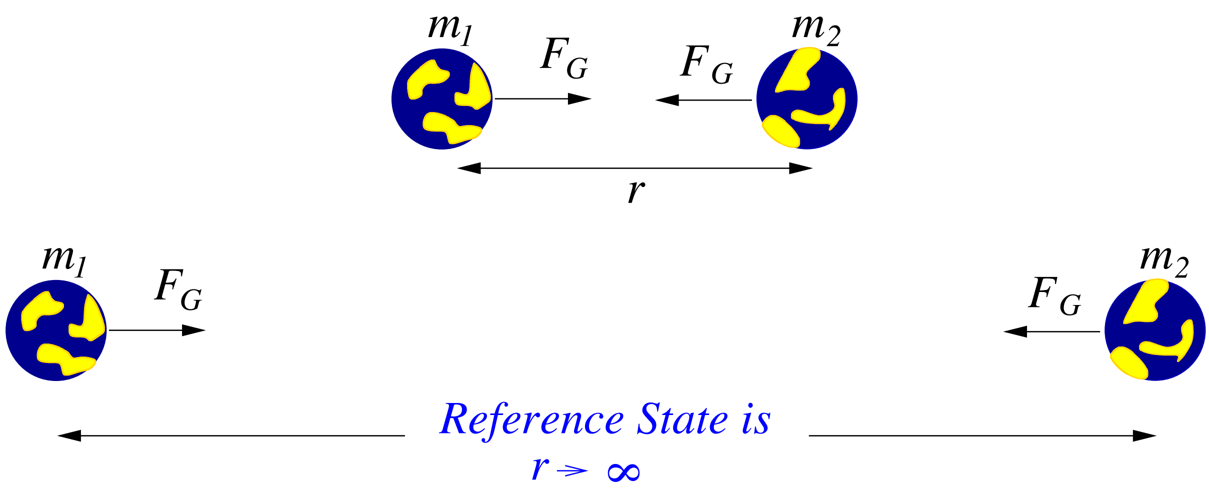
> KEY POINT **3.13**
> **Gravitational potential energy.** The potential energy of a body at height h above some reference point (say the earth's surface) is
> $$U = mgh$$

Note: the reference point we choose is arbitrary ← ANALYSIS

[C] Gravitational potential energy of two point masses

- The gravitational force acting between two masses is proportional to each mass, and inversely proportional to the square of their separation (KP 2.4)

> KEY POINT **3.14**
> **Gravitational Potential Energy.** The gravitational potential energy of a particle of mass m_2 a distance r from a body of mass m_1 is
> $$U = -\frac{Gm_1m_2}{r}$$
> where G is the gravitational constant: 6.673 $\times$ $10^{-11}Nm^2kg^{-2}$



← ANALYSIS
← T
← T

Learning Resources

- **Textbook:** HRW Chapter 8.4
- **Self-Test Questions:** available on–line
- **Course Questions:** Ski Lift, Gravitational escape energy.

S3.7 Energy conservation

[A] Conservation laws in physics

- If the total amount of some quantity, Q say, does not change with time, we say it is conserved. Mathematically, we can express this as

$$\frac{dQ}{dt} = 0$$

- Conservation laws are fundamental laws of nature. ← IS THAT CLEAR?

- Conservation laws are useful because:
  - knowing a quantity is conserved is a very valuable aid to problem solving.
  - they provide restrictions on the form of hypotheses that can be advanced.

[B] Energy Conservation

Energy cannot magically appear or disappear.

KEY POINT **3.15**
Energy Conservation. In a closed system, energy is conserved; it can only be transformed between one form and another.

- A closed system is one in which there is no mass or energy flow across its boundaries.

- It is believed that this conservation law is a fundamental law of nature. ← I

KEY POINT **3.16**
Conservation of Mechanical Energy. In a closed system in which only conservative forces act, then

$$K_i + U_i = K_f + U_f$$

where K_i, K_f, U_i and U_f are the initial and final kinetic and potential energies.

This law is useful because it tells us something that does not change, even when various types of energy are changing.

- The connections between kinetic energy, potential energy and energy conservation can be derived very elegantly. ← M

← DEMOS

← WORKED EXAMPLE

← A

[C] Violation of energy conservation?

- In some cases it may appear that energy conservation is violated. In fact this is not the case; what is happening is that energy is being transformed into a form that has not been accounted for e.g. heat, sound, electrical energy, strain energy, chemical energy. Often, this transformed energy is difficult to measure.

- One remedy is to reconsider what constitutes our 'system' in such a way as to account for the 'lost' energy. In doing this it is sometimes advantageous to introduce phenomological forces such as friction (S2.8) (which accounts for energy transformed into heat.)

- In assessing matters of energy conservation, one has to decide when and whether it is safe to ignore energy transformed into other hard-to-measure forms such as heat. In tutorial problems, you will often be given clues, or just told that energy is conserved.

← WORKED EXAMPLE

← T

Learning Resources

- **Textbook:** HRW Chapter 8.5, 8.8
- **Self-Test Questions:** available on–line
- **Course Questions:** Kinetic and potential energy, Energy conservation .... in disguise, Lift Off, Balance, Non-conservative forces, Dissipation.

S4 Linear Momentum

Motivation: The concept of the linear momentum of a system of particles is an extremely fruitful one in many areas of physics. In this section we develop the tools needed to describe the motion of such a system, and deduce that momentum must be conserved for an isolated system. This allows us to analyse elastic and inelastic collisions.

Objectives: By the end of this section you should

- be familiar with Newton's 2nd law in its most general form
- understand what is meant by a 'system of particles'
- be able to calculate the centre of mass, its velocity and momentum
- understand why, for a system of particles, linear momentum is conserved
- be able to analyse inelastic collisions by applying the principle of momentum conservation
- be able to analyse elastic collisions using momentum and kinetic energy conservation

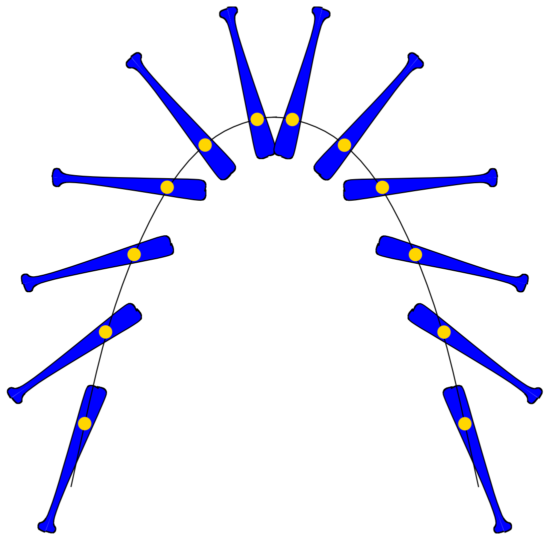
Key Mathematics: Vectors in one and two dimensions

S4.1 Preview

[A] The motion of a rigid body

When a rigid body is thrown into the air, we notice something simple in its apparently complex motion: one special point in the object moves as though

- all the mass were concentrated at that point
- the external force (gravity in this case) acts only at that point



This special point, the center of mass, forms the starting point for our study of the motion of collections (or systems) of particles, of which a rigid body is a special case.

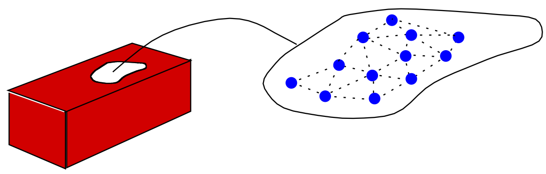
By analysing the dynamics of the center of mass we discover a new fundamental law of nature: the law of conservation of momentum. We use this new law to analyse collisions between different bodies, which can be elastic or inelastic. ← DEMO

S4.2 Systems of particles

[A] What is a system of particles?

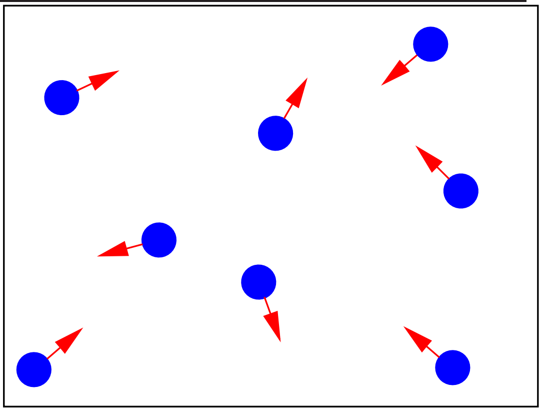
A system of particles is any set of particles whose properties we wish to consider collectively.
For example:

• We can think of a rigid object as being made up of many particles, with fixed relative positions.



Or alternatively,

• a collection of gas molecules contained in a box is a system of particles



← NOTE

[B] Centre of Mass

KEY POINT **4.1**
The Centre of Mass of a system of particles with positions $\vec{r}_i$ is defined as

$$\begin{aligned}\vec{r}_{cm} &= \frac{1}{M}\left[m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots\; m_n\vec{r}_n\right] \\ &= \frac{1}{M}\sum_i m_i\vec{r}_i\end{aligned}$$

where the total mass of the system is

$$M = \sum_i m_i$$

We can easily check some special cases:

- One particle: the centre of mass is

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1}{m_1} = \vec{r}_1.$$

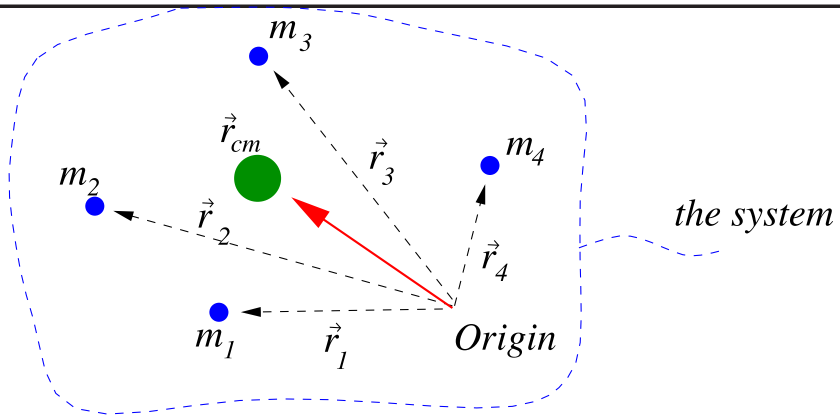
- Two particles of equal mass: the centre of mass is

$$\vec{r}_{cm} = \frac{m\vec{r}_1 + m\vec{r}_2}{m+m} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

  i.e. halfway between the two particles.

- Many particles of equal mass: the centre of mass is the average position of all of the particles.

  You can think of the centre of mass as being a (mass) weighted average of the particles positions.



← COMMENTARY
← WORKED EXAMPLE
← COMMENTARY
← DEMO
← WORKED EXAMPLE
← I
← I
← M
← T

Learning Resources

- **Textbook:** HRW Chapter 9.1, 9.2. Ignore the material on calculating the centre of mass for continuous objects by integration. This is a straightforward, but tedious, extension of our definition from which you learn little apart from integration.
- **Self-Test Questions:** available on–line
- **Course Questions:** Calculating the centre of mass I, Calculating the centre of mass II.

S4.3 Motion of the centre of mass

[A] Centre of mass and Newton's 2nd law

- Let's begin by writing the centre of mass vector (KP 4.1) in the following way:

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... m_n\vec{r}_n \tag{4.1}$$

- The centre of mass will evolve with time. Its velocity is defined (KP 1.1) by differentiating Equation 4.1 term-by-term with respect to time.

KEY POINT **4.2**
Velocity of the centre of mass:

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... m_n\vec{v}_n$$

- The acceleration of the centre of mass is given by another differentiation (KP 1.2)

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + ... m_n\vec{a}_n$$

- Using Newton's 2nd law (KP 2.2) we can write this as

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \ldots \vec{F}_n \tag{4.2}$$

where $\vec{F}_1$ is the force acting on particle 1, etc.

- The forces $\vec{F}_1$, $\vec{F}_2$, etc, acting on the particles are of two types:
  - those acting on the system from outside: external forces
  - forces acting within the system: internal forces
- By Newton's 3rd law (KP 2.3), the internal forces form action-reaction pairs and cancel from the sum in Equation 4.2. All that is left are the external forces.

KEY POINT **4.3**
Newton's 2nd law for a system of particles. The motion of the centre of mass is governed by external forces only

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

- This gives us another way of viewing the centre of mass:

KEY POINT **4.4**
The centre of mass of a system of particles is that point that moves as though all of the mass were concentrated there and all external forces were applied there.

←A

←M

[B] Consequences

KP 4.3 is the key result. It tells us that:

- if the centre of mass has zero velocity and no external forces act, then regardless of the motion of the individual bodies in the system, the position of the centre of mass remains fixed (Making use of the centre of mass II).
- linear momentum is conserved. We will learn more about linear momentum in subsequent sections.

←T

Learning Resources

- **Textbook:** HRW Chapter 9.3
- **Self-Test Questions:** available on–line
- **Course Questions:** Making use of the centre of mass I.

S4.4 Linear momentum

[A] Definition

KEY POINT **4.5**
The **Linear Momentum** of a single particle is defined as the product of the particle's mass m, and its velocity $\vec{v}$:

$$\vec{p} = m\vec{v}$$

Note: this is a vector equation.

[B] Newton's 2nd Law

KEY POINT **4.6**
Newton's 2nd Law is most economically expressed in terms of the momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

- This is identical to the previous definition (KP 2.2), provided the mass of the particle is constant

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

- This form of the 2nd law is useful when the mass of a body changes with time, for instance when a rocket takes off. $\leftarrow\boxed{1}$

[C] Integral form

When a force acts on a body, the momentum gained by the body is just the sum (or integral) of all the forces acting over time. This allows us to define an average force in terms of the momentum.

$$\int_0^t \vec{F}(t)dt = \vec{p}(t) - \vec{p}(0) = \vec{\Delta p}$$

KEY POINT **4.7**
The average force acting on a body over time Δt producing momentum change $\vec{\Delta p}$ is defined as:

$$\vec{F}_{av}\Delta t = \vec{\Delta p}$$

WORKED EXAMPLE
T

Learning Resources

- **Textbook:** HRW Chapter 9.4, 9.5
- **Self-Test Questions:** available on–line
- **Course Questions:** Kinetic energy and linear momentum.

S4.5 Linear momentum and its conservation

[A] Linear momentum of a system of particles

Using the velocity of the centre of mass (KP 4.2), and the definition of linear momentum for a single particle (KP 4.5), we can identify the total linear momentum of the system $\vec{P}_{tot}$ as follows:

KEY POINT **4.8**
Linear momentum of centre of mass is defined as

$$\vec{P}_{tot} = M\vec{v}_{cm} = \vec{p}_1 + \vec{p}_2 + ... \vec{p}_n$$

[B] Newton's 2nd law

We can now express Newton's 2nd law in terms of the momentum.

KEY POINT **4.9**
Newton's 2nd law for a system of particles is:

$$\vec{F}_{ext} = \frac{d\vec{P}_{tot}}{dt}$$

WORKED EXAMPLE

[C] Conservation of Linear Momentum

If the total external force acting on a system is zero then we can see that the total linear momentum of the system must be a constant. This is the law of conservation of linear momentum.

KEY POINT **4.10**
Conservation of Linear Momentum: if a system of particles is isolated from its surroundings then linear momentum is conserved:

$$\frac{d\vec{P}_{tot}}{dt} = 0 \Rightarrow \vec{P}_{tot} = \text{constant}$$

Note: Since linear momentum is a vector quantity, each component of linear momentum is conserved separately. If a component of the net external force on a system is zero along an axis, then the component of linear momentum of the system along that axis cannot change.

[D] Isolated systems

Saying a system is isolated is really another way of saying that the total external force is zero; hence that linear momentum is conserved within the system.
Examples of isolated (or approximately isolated) systems:

- A star/brick/collection of atoms in free space.
- A car sliding on very slippy ice (provided the ice is very slippy and air resistance is ignored).
- A railway truck on horizontal rails (provided the wheels are well oiled, and rolling resistance is very small).

When trying to apply the law of conservation of linear momentum, you should make sure the system you choose is isolated (at least in the direction you want). Additionally, if you want to apply the law of conservation of energy, the system must be closed (no particles enter or leave the system). ← WORKED EXAMPLE

Learning Resources

- **Textbook:** HRW Chapter 9.7
- **Self-Test Questions:** available on–line
- **Course Questions:** Making use of the centre of mass II, Wait for weight.

S4.6 Collisions

[A] Context

- A collision occurs whenever two bodies interact strongly for a short time.
- Collisions can be completely elastic, completely inelastic, or somewhere in between. ← D

← D

- Conservation of linear momentum and where appropriate energy can be very powerful techniques for analysing collisions.

←A

←EXAMPLES

←COMMENTARY

[B] Impulse

- In a collision, forces can vary rapidly as a function of time. We use impulse to quantify the strength and duration of a collision.

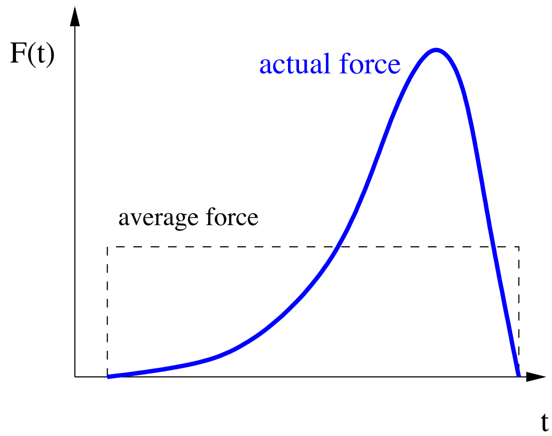
KEY POINT **4.11**
Impulse is the change in momentum a particle experiences during a collision

$$\vec{I} = \int_0^t \vec{F}(t)dt = \vec{\Delta p}$$

By definition, the impulse is equal to the average force (KP 4.7) acting during the collision multiplied by the duration of the collision:

$$\vec{I} = \vec{F}_{av}\Delta t$$

For a $1D$ collision, the impulse is the area under the $F(t)$ curve describing the collision



←DEMO

[C] Inelastic collisions

- Collisions in which energy is dissipated are termed inelastic ←D

- The collision of two cars, with the subsequent tearing and deforming of metal, plastic and organic matter dissipates a lot of energy, in work done on the metal, in noise and in heat: car crashes are definitely inelastic.

[D] Elastic collisions

- Collisions in which no energy is dissipated are termed elastic. ←D

- Collisions are elastic, if, for a closed system of two particles the total kinetic energy (in addition to the momentum) is conserved. Using the same notation as above, we have an additional relation expressing conservation of kinetic energy.

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \tag{4.3}$$

- the collision of a hard rubber ball with the floor is almost completely elastic. The ball rises almost to the height from which it was released.

←DEMO

←A

[E] Problem solving tactics for collisions

- Decide on the boundaries of your system.

- Decide if your system is closed and isolated. Closed means no matter passes through the boundaries. Isolated means interactions only occur between particles in your system: the net external force is zero.

- Decide if energy is conserved. In most problems you will meet it will not be.

- Remember that in a closed, isolated system, linear momentum is conserved regardless of whether the collision is elastic or inelastic. If particles 1 and 2 have momenta $\vec{p}_1$ and $\vec{p}_2$ respectively, momentum conservation implies that

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \tag{4.4}$$

  where additional subscripts i and f denote initial and final momenta.

- Remember also that linear momentum will be conserved for each of its components. (If the system is not isolated it may be the case that momentum will be conserved in one component only, say the component perpendicular to an external force.)

- Select two states before and after a collision, and equate momenta (and where appropriate energies) before and after the collision. Solve for what is required.

←WORKED EXAMPLE

←T

Learning Resources

- **Textbook:** HRW 9.6 and 9.8-9.11
- **Self-Test Questions:** available on–line
- **Course Questions:** A simple collision, Two spheres colliding, Road Accident, A reprise on problem solving.

S4.7 Relativity: Mass, Momentum and Energy

This section is included for general interest. It is not part of the examinable programme of the course.

Learning Resources

- **Textbook:** The presentation here is taken mainly from 'Special Relativity' by A.P. French (Van Nostrand Reinhold (UK) Ltd: Wokingham)

S5 Angular Momentum

Motivation: In this section we develop concise methods of describing rotational motion using quantities such as angular velocity, angular momentum and moment of inertia, the rotational analogues of velocity, momentum and mass. Using this new language, we can describe such counterintuitive phenomena as the behaviour of spinning tops and gyroscopes, and find out why it is easier to ride a bicycle with bigger wheels.

Objectives: By the end of this section you should

- know the analogies between linear and rotational motion
- understand the concepts of angular velocity, angular momentum and torque, and their connection to Newton's second law expressed in angular variables
- be able to use conservation of angular momentum to describe simple physical phenomena.

Key Mathematics: Vector cross products

S5.1 Linear and rotational motion

[A] Analogies between linear and rotational motion

To describe rotational motion we use angular variables. These are defined so that Newton's laws take on their familiar forms when expressed in angular form. The rotational analogues of linear variables and physical laws are listed below.

KEY POINT **5.1**

Linear Motion				Angular Motion
position	x	$\longleftrightarrow$	θ	angle
velocity	$\vec{v}$	$\longleftrightarrow$	$\vec{\omega}$	angular velocity
acceleration	$\vec{a}$	$\longleftrightarrow$	$\vec{\alpha}$	angular acceleration
mass	m	$\longleftrightarrow$	I	moment of inertia
momentum	$\vec{p}$	$\longleftrightarrow$	$\vec{L}$	angular momentum
force	$\vec{F}$	$\longleftrightarrow$	$\vec{\tau}$	torque
	$\vec{F} = m\vec{a}$	$\longleftrightarrow$	$\vec{\tau} = I\vec{\alpha}$	
	$\vec{p} = m\vec{v}$	$\longleftrightarrow$	$\vec{L} = I\vec{\omega}$	
K.E.	$K = \frac{1}{2}mv^2$	$\longleftrightarrow$	$K_R = \frac{1}{2}I\omega^2$	Rotational K.E.

These equations are more than a convenient rewriting of Newton's laws. They contain new physics, in particular the law of conservation of angular momentum. During this module, we will explore these analogues, and their consequences.

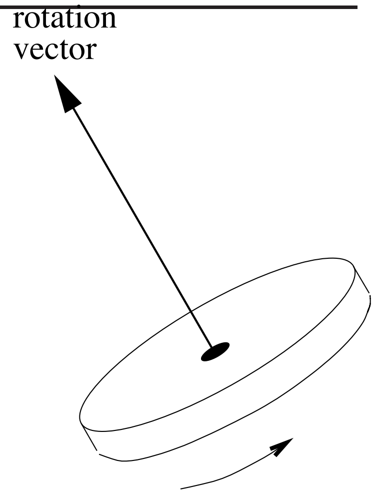
Be warned: they can be very counterintuitive!

S5.2 Angular positions, velocities and accelerations

[A] Using vectors to describe rotations

- To describe the rotation of a rigid body, we need to specify
  - the axis of rotation
  - the sense of the rotation (clockwise/anti-clockwise)
  - the magnitude of the rotation.
- We can describe a rotation using a rotation vector:
  - the magnitude of the axial vector is given by the rotation angle in radians.
  - the direction of the rotation vector is given by the axis of rotation in conjunction with the corkscrew rule

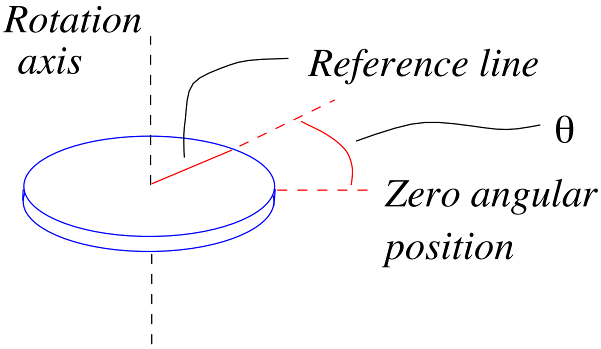
Corkscrew rule: Take a corkscrew and align it with the axis of rotation. Turn it in the sense of the rotation.
The direction it moves defines the direction of the rotation vector.



- Under a rotation every point on the body moves in a circle centred on the rotation axis.

[B] Angular position

- To define the angular position of a rigid body, we need to specify the rotation axis, and an origin. We choose the origin as follows:
  - we draw an imaginary line in the rigid body, which will rotate with the body
  - we draw a line outside the body, which will remain fixed.
  - the angular position of the body can then be defined as the angle, θ, the line in the body makes with the line fixed outside the body.



- The definitions of angular displacement, velocity and acceleration can now be completed in a straightforward manner.

← COMMENTARY

[C] Angular displacement

KEY POINT 5.2
Angular Displacement: If a rigid body rotates about a fixed axis changing the angular position from θ_1 to θ_2, the body undergoes an angular displacement

$$\Delta\theta = \theta_2 - \theta_1$$

Note: angular displacement is measured in radians.

[D] Angular velocity

KEY POINT 5.3
Average angular velocity: If a rigid body rotates around a fixed axis producing an angular displacement $\Delta\theta$ in time Δt, the average angular velocity about the axis is

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

KEY POINT 5.4
Instantaneous angular velocity is defined as the average angular velocity over the next infinitesimal time interval

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

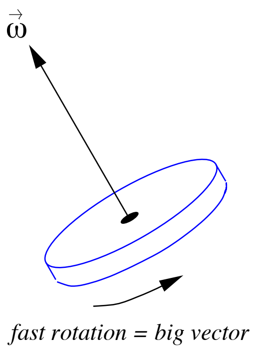
Note: the units of angular velocity are $rad\, s^{-1}$.

← NOTE

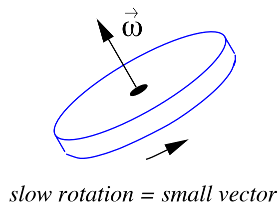
- Note: strictly speaking, what we have defined above are angular speeds, not velocities, since we have not specified the direction of rotation.

- We can define an angular velocity vector, $\vec{\omega}$. The axis of rotation determines the direction of the angular velocity vector; the magnitude of the vector is the angular velocity ω as defined above and the sense of rotation is specified by the corkscrew rule.

A large angular velocity vector



A small angular velocity vector



←WORKED EXAMPLE

[E] Angular acceleration

KEY POINT **5.5**
Average angular acceleration: If a rigid body accelerates about a fixed axis producing a change in angular velocity $\Delta\omega$ in time Δt the average angular acceleration is

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$$

KEY POINT **5.6**
Instantaneous angular acceleration is defined as the average angular acceleration over the next infinitesimal time interval

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Note: the units of angular acceleration are $rad\, s^{-2}$.

- We can define an angular acceleration vector, $\vec{\alpha}$. The rotation axis specifies the direction of the angular acceleration vector; the magnitude is the angular acceleration α as defined above; the sense is given by the corkscrew rule.

←WORKED EXAMPLES

Learning Resources

- **Textbook:** HRW Chapter 10.1-10.3
- **Self-Test Questions:** available on–line
- **Course Questions:** Rotating wheel, Visualising angular velocities, Injuns!.

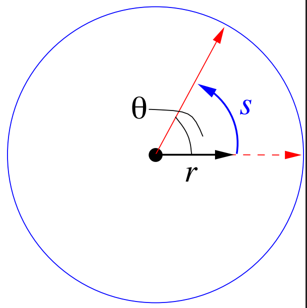
S5.3 Relations between angular and linear quantities

[A] Position

KEY POINT **5.7**

If a body is rotated through an angle θ, a point a distance r from the rotation axis is moved in a circular arc of length

$$s = r\theta$$



COMMENTARY

[B] Speed and period

KEY POINT **5.8**
The speed, v of a point a distance r from the rotation axis of a body with angular velocity ω is

$$v = r\frac{d\theta}{dt} = r\omega$$

The period of revolution is given by [distance]/[velocity]

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

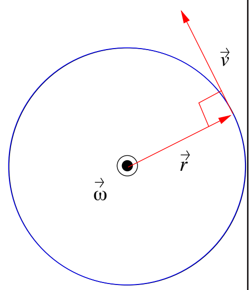
WORKED EXAMPLE

[C] Velocity

KEY POINT **5.9**

The velocity of a point $\vec{r}$ from the rotation axis is related to the angular velocity vector, $\vec{\omega}$ by the cross product (KP 0.4)

$$\vec{v} = \vec{\omega} \times \vec{r}$$



($\vec{\omega}$ points out of the page)

[D] Acceleration

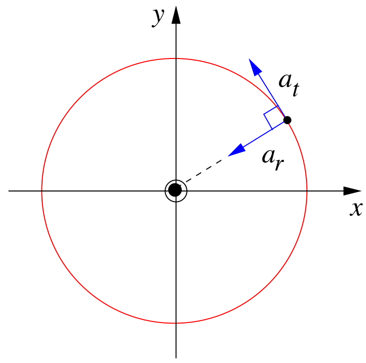
KEY POINT **5.10**

The tangential component of the (linear) acceleration is

$$a_t = r\frac{d\omega}{dt} = r\alpha.$$

The radial component of the (linear) acceleration, a_r (KP 1.8) is

$$a_r = \frac{v^2}{r} = r\omega^2$$



($\vec{\alpha}$ points out of the page)

COMMENTARY
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Learning Resources

- **Textbook:** HRW Chapter 10.5
- **Self-Test Questions:** available on–line
- **Course Questions:** Angular velocity.

S5.4 Constant acceleration equations

[A] Rotation with constant acceleration

Consider the motion of a body rotating about a fixed axis, with constant acceleration.

- Since the rotation axis is fixed, displacement, velocity and acceleration vectors all point in the same direction. The motion is, in a sense, one dimensional.

- The relationships between displacement, velocity and accleration (KP 5.4 and KP 5.6) are

$$\omega = \frac{d\theta}{dt}; \qquad \alpha = \frac{d\omega}{dt}$$

- By analogy with the relations describing one dimensional linear motion (KP 1.1 and KP 1.2), we can immediately write down the constant acceleration equations (c.f. KP 1.4)

KEY POINT **5.11**
Constant Acceleration Equations:

$$\omega = \omega_0 + \alpha t \quad (a) \qquad \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \quad (b) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad (c)$$

← WORKED EXAMPLE

Learning Resources

- **Textbook:** HRW Chapter 10.4
- **Self-Test Questions:** available on–line
- **Course Questions:** Angular acceleration.

S5.5 Kinetic energy of a rotating body: moment of inertia

- A rotating body stores kinetic energy.
- The amount of kinetic energy stored depends on the angular velocity and the distribution of mass around the rotation axis - the moment of inertia.
- The moment of inertia is in general different for rotations about different axes.
- Let's calculate the kinetic energy of a rigid rotating body composed of n particles, mass m_i, located a distance r_i from the rotation axis

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \left(r_i^2 \omega^2\right) = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2$$

- This defines the moment of inertia, which is the rotational equivalent of mass.

KEY POINT **5.12**
Moment of inertia for a rigid body composed of n particles mass m_i a distance r_i from the rotation axis is

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Note: the moment of inertia depends on the rotation axis

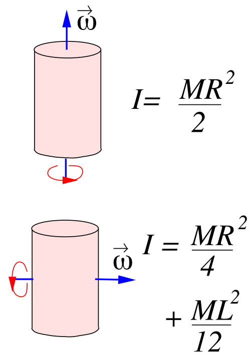
- The units of moment of inertia are $kg\, m^2$.

- Using KP 5.8 we can rewrite the rotational kinetic energy of a body in a form reminiscent of KP 3.8

> KEY POINT **5.13**
> **Rotational kinetic energy** of a rigid body rotating about a fixed axis is
>
> $$K = \frac{1}{2} I \omega^2$$
>
> where I is the moment of inertia about that axis.

← COMMENTARY

Example: the moment of inertia of a solid cylinder, radius R, length L, mass M about two different axes.
Note: for different rotation axes, we get different I's.



← WORKED EXAMPLE

← DEMO

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Learning Resources

- **Textbook:** HRW Chapter 10.6 and 10.10. Ignore section 10.7 on calculating moments of inertia by integration.
- **Self-Test Questions:** available on–line
- **Course Questions:** Translational and rotational motion.

S5.6 Torque

[A] Torque

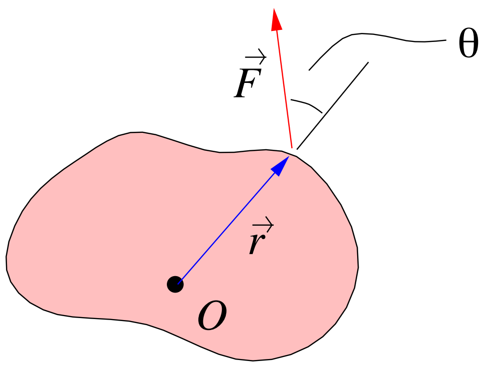
- A **torque** is a vector that describes the ability of a force to change the angular velocity of a body.

KEY POINT **5.14**

Torque. Applying a force, magnitude F to a point a distance r from point O, results in a torque about O of magnitude

$$\tau = rF\sin\theta.$$

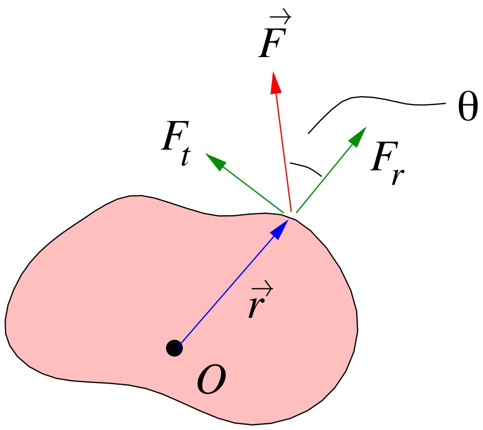
The units of torque are $N\,m$.



There are two equivalent ways of computing the torque:

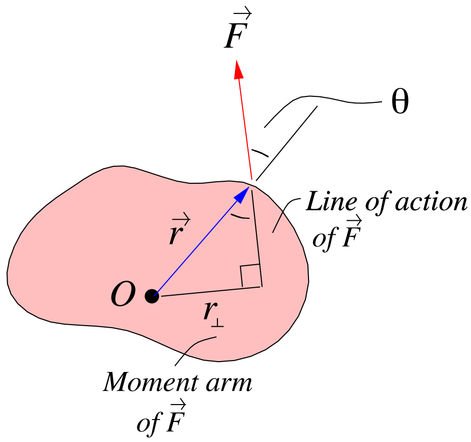
- Multiplying the tangential component of the applied force, F_t, by the distance from the rotation axis

$$\tau = rF_t.$$



- Multiplying the applied force by $r_\perp$, the perpendicular distance between the rotation axis and an extended line running through the vector F (the line of action of $\vec{F}$).

$$\tau = r_\perp F$$

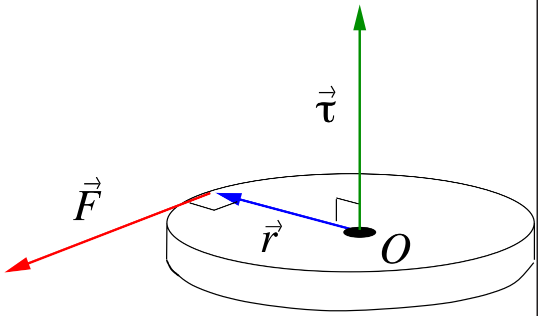


[B] The torque vector

KEY POINT **5.15**
Torque vector: The torque about a point O is the vector defined by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the point at which the force $\vec{F}$ is applied with respect to point O.



WORKED EXAMPLE

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Learning Resources

- **Textbook:** HRW Chapter 10.8 and 11.6
- **Self-Test Questions:** available on–line
- **Course Questions:** Torque, Torque and angular acceleration, Slowing flywheel.

S5.7 Angular momentum

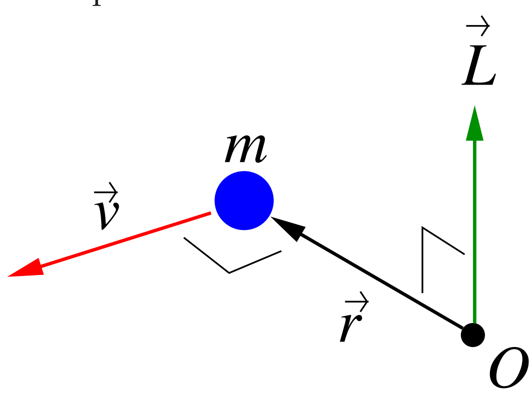
[A] Definition

- Like all other linear quantities, linear momentum has its angular counterpart: angular momentum.
- Angular momentum is a vector with units $kg\, m^2\, s^{-1}$.

KEY POINT **5.16**
The angular momentum of a particle with respect to a point O is

$$\vec{L} = \vec{r} \times \vec{p} = m\,(\vec{r} \times \vec{v})$$

$\vec{r}$ is the position of the particle with respect to O.



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[B] Angular momentum of a rigid body

KEY POINT **5.17**
The magnitude of the angular momentum of a rigid body rotating about an axis with angular velocity ω is

$$|\vec{L}| = I\omega$$

where I is the moment of inertia (KP 5.12) about the rotation axis. The direction of $\vec{L}$ is along the axis in the sense of the rotation vector.
Note: the moment of inertia depends on the rotation axis.

[C] Newton's 2nd law for rotation

- It is possible to formulate Newton's 2nd law in rotational form.
- By considering the total angular momentum of a set of particles about a given point, and using Newton's third law (KP 2.3) to eliminate internal forces, one obtains the general expression relating torque and angular momentum.

KEY POINT **5.18**
Newton's 2nd Law: angular form relates the total external torque, $\vec{\tau}_{ext}$, acting on a system of particles, to the total angular momentum $\vec{L}$ of the system about the same point.

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

$\vec{\tau}$ and $\vec{L}$ are expressed with respect to the same origin.

[D] Special case: Newton's 2nd law for rigid bodies

For the special case of a rigid body rotating about an axis, Newton's second law takes a particular form involving the moment of inertia.

KEY POINT **5.19**
Newton's 2nd law for rigid bodies: A torque, τ, applied to a rigid body rotating about a fixed axis produces an angular acceleration α about that axis

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

where I is the moment of inertia about the rotation axis, ω the angular velocity and θ the angular displacement.

Learning Resources

- **Textbook:** HRW Chapter 10.9. 11.7-11.10
- **Self-Test Questions:** available on–line
- **Course Questions:** Spot the difference, Angular momentum.

S5.8 Angular momentum conservation

[A] Conservation law

Given KP 5.18, in the absence of any external torques, we conclude that angular momentum is conserved.

KEY POINT **5.20**
Angular momentum conservation. If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

← WORKED EXAMPLE

← DEMO

[B] Consequences

There are many profound consequences of the law of conservation of angular momentum. In particular it explains:

- how gyroscopes behave;
- why it's easier to ride a bicycle with big wheels;
- how gymnasts, divers and trapeze artists somersault;
- the shape of the solar system;
- why neutron stars and black holes rotate so fast.
- children's toys !

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Learning Resources

- **Textbook:** HRW Chapter 11.11-11.12 has many good examples.
- **Self-Test Questions:** available on–line
- **Course Questions:** Stop the earth, I want to get off!, Torque and angular momentum, Skaters, Roundabout.

S6 Oscillations

Motivation: Understanding and exploiting oscillations and waves is central to many aspects of science including physics, chemistry, biology and engineering. In this section we will set out the key concepts, and explore them in the context of a wide range of examples. We will end up in chaos.

Objectives: By the end of this section you should

- be familiar with the key concepts and terms used to describe oscillatory and wavelike phenomena
- be able to set up the simple harmonic equation of motion for a range of systems and determine the associated SHM frequency
- be familiar with the mathematical description of sinusoidal waveforms and the variety of phenomena occurring when different waveforms are combined

Key Mathematics: sines, cosines and differential equations

S6.1 Introduction: what and why

- Oscillations are repetitive, often regular, changes of some ’thing’
- Waves are evident in the behaviour of many individual oscillating ’things’.
- These phenomena are widespread in science and technology. ←I
- They are characterised by an enormous range of time and length scales. ←I ←I
- But this diversity is underpinned by a single set of tools (concepts and supporting mathematics) ←I ←I
- The tools ’empower’ (give us generalisable understanding) and will be central here. ←I ←I
- We focus on ‘oscillations’; the extension to ‘waves’ follows in Physics 1B ←I

←I ←I ←I

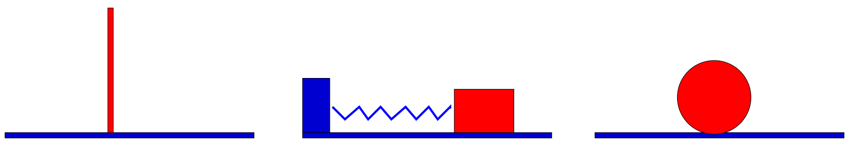
Learning Resources

- **Textbook:** HRW 15.1-15.2 ←EXAMPLES

S6.2 Simple Harmonic Motion: the physical context

[A] Kinds of equilibrium

- A body subject to no net external force or torque, and initially at rest, will remain so (KP 2.1): it is in static equilibrium.
- We can usefully distinguish between three kinds of static equilibrium which differ according to the forces called into play when the body is displaced a little from the equilibrium position.

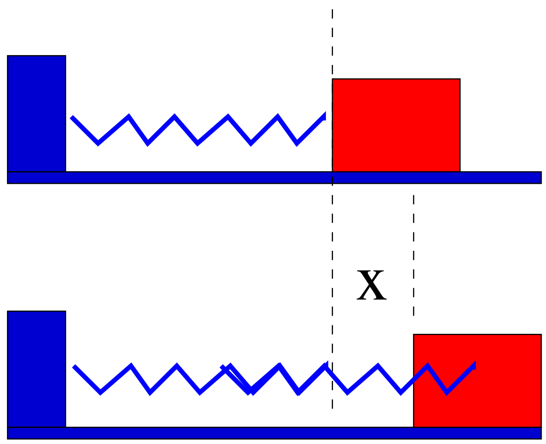


(a) (b) (c)

  - In unstable equilibrium (figure (a)) the forces drive the body further away from the equilibrium position.
  - In stable equilibrium (figure (b)) the forces drive the body back towards the equilibrium position.
  - In neutral equilibrium (figure (c)) the body continues to experience no net force.
- We are concerned entirely with systems near a point of stable equilibrium.

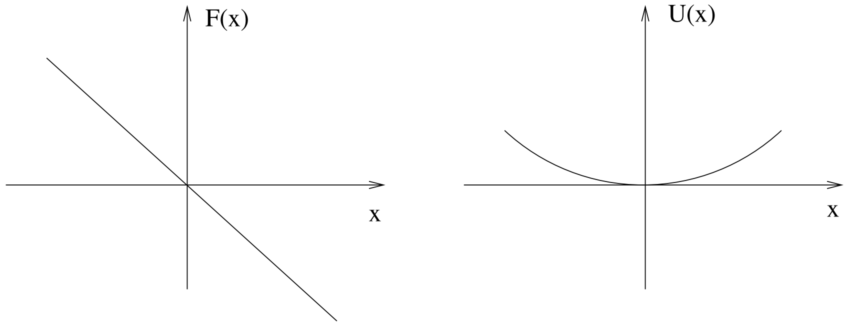
[B] What happens near stable equilibrium

- Consider the behaviour of a system near a position of stable equilibrium.
  - To be specific, we will choose the mass-spring system (figure (b) above; and right)
  - But we will then see that our arguments are general.



- We can describe the behaviour in two entirely equivalent ways:
  - in the language of forces

- in the language of potential energies



- The force called into play is linear and restoring (S2.9):

$$F(x) = -kx$$

with k a positive constant.

- The associated potential energy is given by

$$U(x) = U(0) + \frac{1}{2}kx^2$$

where $U(0)$ is a constant (frequently set to zero).

- The two functions are related by KP 3.11:

$$F = -\frac{dU}{dx}$$

- The fact that $x = 0$ is a point of equilibrium implies that the force is zero there. Hence

$$\frac{dU}{dx} \mid_{x=0} = -F(x=0) = 0 \tag{6.1}$$

so $U(x)$ has a turning point at $x = 0$.

- The fact that $x = 0$ is a point of stable equilibrium implies that the force is restoring ie that $k > 0$. Hence

$$\frac{d^2U}{dx^2} \mid_{x=0} = -\frac{dF}{dx} \mid_{x=0} = k > 0 \tag{6.2}$$

implying that the turning point is a minimum.

- These results hold true more generally:

KEY POINT **6.1**
The behaviour of a body given a small displacement from a point of stable equilibrium can be described, equivalently, by a force $F(x)$ and a potential energy $U(x)$ with the forms

$$F(x) = -kx \text{ and } U(x) = U(0) + \frac{1}{2}kx^2$$

with $k > 0$. The point $x = 0$ is a minimum of the potential energy.

COMMENTARY

[C] SHM: the key equation

- Again consider the specific case of the mass-spring system.
- Suppose that (at some instant) the mass, m say, has some displacement x.
- Its acceleration at that instant follows from Newton's 2nd Law (KP 2.2)

$$ma = m\frac{d^2x}{dt^2} = m\ddot{x} = F(x) = -kx$$

  where we have assumed that the only force the mass experiences is that due to the spring.
- We will write this equation in the form

$$\ddot{x} = -\omega^2 x \tag{6.3}$$

  where

$$\omega \equiv \sqrt{\frac{k}{m}} \tag{6.4}$$

- At this point ω is a convenient abbreviation; note that its units are s^{-1}.
- Equation 6.3 is the equation of motion for the mass: it expresses the acceleration at an instant in terms of the displacement at that instant.
- The motion that emerges from this equation
  is known as simple harmonic motion (SHM).
- We may generalise as follows:

KEY POINT **6.2**
The equation of motion of a system near to a point of stable equilibrium is of the form

$$\ddot{x} = -\omega^2 x$$

where ω is a constant, characteristic of the system. This is the fundamental equation of simple harmonic motion. A system obeying this equation is described as a simple harmonic oscillator.

← COMMENTARY

Learning Resources

- **Textbook:** HRW section 15.2-15.3
- **Self-Test Questions:** available on–line
- **Course Questions:** Kinds of equilibrium, Taylor series expansion.

S6.3 The SHM equation: a general tour

← PREAMBLE

[A] The solution: algebra

- The mathematical problem is to solve the SHM equation (KP 6.2).
- This is a differential equation; solving it means finding the general form of a function $x(t)$ which satisfies it.
- Not all differential equations are solvable in this way; but this one is.
- It is solvable (easily) because it is linear:

KEY POINT **6.3**
The general solution to the SHM equation $\ddot{x} = -\omega^2 x$ is

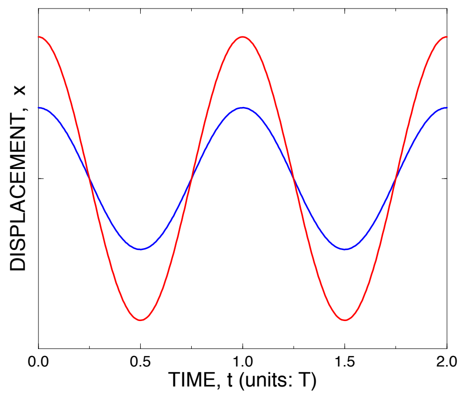
$$x(t) = x_m \cos(\omega t + \phi)$$

where x_m and ϕ are constants determined by the initial conditions.

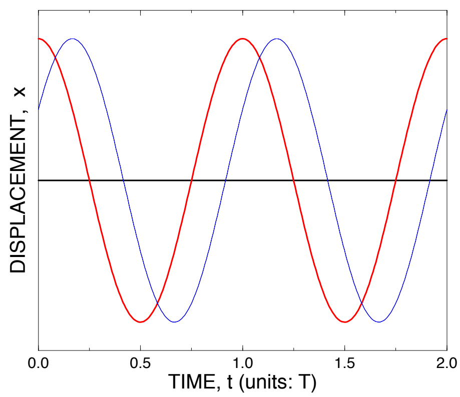
← COMMENTARY

[B] Visualisation: the role of amplitude and phase

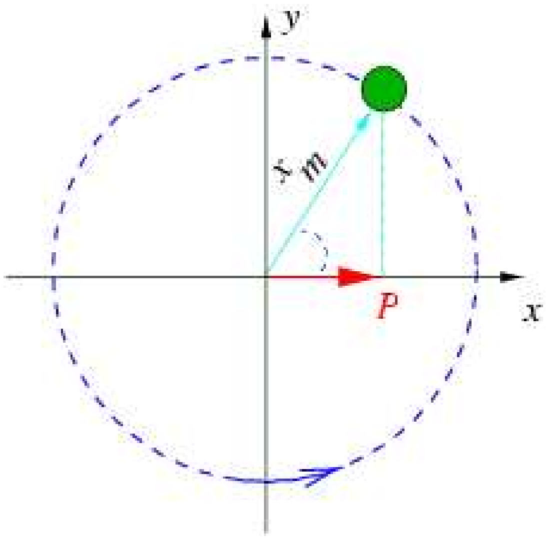
- x_m is called the amplitude of the motion
- Since the cosine function varies between ± 1 the value of x varies between $\pm x_m$
- The figure shows $x(t)$ for two values of x_m, with one twice the other.
- The 'natural' unit of time is the period
$$T = 2\pi/\omega$$



- ϕ is called the initial phase of the motion, or the phase constant.
- We can write
$$\omega t + \phi = \omega(t + \phi/\omega)$$
- Thus a solution with a given ϕ is identical to a zero-ϕ solution shifted bodily along the time axis.
- The figure shows $x(t)$ for two values of ϕ differing by $\phi = 2\pi/6$.



- There is another useful way of visualising the solution to the SHM equation and the role of x_m and ϕ.
- Think of a particle moving with constant angular velocity ω anticlockwise in a circle of radius x_m
- Focus on the radial vector from the centre to the particle.



- Suppose that at $t = 0$ this vector makes an angle ϕ with the x-axis.
- Then at time t this angle is $\omega t + \phi$

- The x coordinate of the particle is the projection of the radial vector on the x axis:

$$x = x_m \cos(\omega t + \phi)$$

which is the SHM solution.

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[C] Visualisation: displacement, velocity and acceleration

- The accompanying figure shows the behaviour of

  - the displacement:

    $$x(t) = x_m \cos(\omega t + \phi)$$

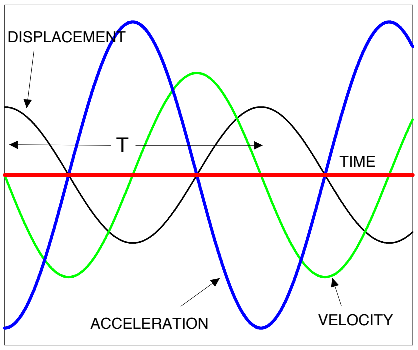
  - the velocity:

    $$v(t) = \dot{x} = -\omega x_m \sin(\omega t + \phi)$$

  - the acceleration:

    $$a(t) = \ddot{x} = -\omega^2 x_m \cos(\omega t + \phi) = -\omega^2 x(t)$$

  on the same time-axis.



- At the equilibrium position, $x = 0$, the acceleration is zero but the speed is maximal.
- At the extremes of motion $x = \pm x_m$ the speed is zero but the acceleration is maximal.
- The velocity is $\pi/2$ out of phase with the displacement.

[D] How initial conditions fix amplitude and phase

- The initial conditions for any equation of motion are defined by two quantities, the displacement and the velocity.
- These two quantities $x(0)$ and $v(0) = \dot{x}(0)$ together define the amplitude x_m and the initial phase ϕ.
- The relationships are expressed in the equations:

$$\tan\phi = -\frac{v(0)}{\omega x(0)} \quad \text{and} \quad x_m^2 = x(0)^2 + \frac{v(0)^2}{\omega^2} \qquad (6.5)$$

←PROOF

Check It!

- Are the units OK? ☐
- Does it make sense? ☐

[E] All about the 'frequency'

- The physical quantity represented by ω in the above equations is called the angular frequency of the motion.
- The analogy between SHM and circular motion (see above) explains this terminology.
- There are alternative ways of expressing the same thing:
  - The frequency is defined by

$$f = \frac{\omega}{2\pi} \tag{6.6}$$

    and gives the number of cycles completed each second
  - The period is defined by

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \tag{6.7}$$

    and gives the time required to complete one cycle.

← COMMENTARY

- Note the units of the three quantities:

quantity	symbol	units
angular frequency	ω	$rad \cdot s^{-1}$
frequency	f	$s^{-1} \equiv Hz$
period	T	s

- Finally note the fundamental differences
  - x_m and ϕ are fixed by initial conditions –ie how we start the system off (displaced but from rest? with a push from the equilibrium position?)
  - ω and T are fixed by the physical system itself
    – what kind of system it is (spring, pendulum, drum ...)
    – its particular physical parameters (spring constant, length of support, tautness ...)

- In particular T is independent of the amplitude of the motion, x_m ←THINK ABOUT IT!

←T

Learning Resources

- **Textbook:** HRW 15.3
- **Self-Test Questions:** available on–line
- **Course Questions:** The SHM equation, Visualising SHM behaviour, Finding the amplitude and phase: initial conditions, Thinking about SHM, Consolidation exercise.

S6.4 The SHM equation: applications

[A] The task and the strategy

- We have established (S6.2) that if a system in stable equilibrium is given a displacement x the resulting time-evolution of x will follow the SHM equation

$$\ddot{x} = -\omega^2 x$$

where ω follows from the properties of the system.

- We have also established that the general solution to this equation of motion is

$$x(t) = x_m \cos(\omega t + \phi)$$

where x_m and ϕ follow from the initial conditions.

- To apply the general theory to any specific case we are faced with one task: we must determine the SHM frequency.

- The strategy is always the same:
  - We identify an arrangement of stable mechanical equilibrium, whose signature is that it is a minimum of potential energy.
  - We think of what happens if the system is displaced by a small amount x from equilibrium; we usually have a choice as to what x will represent.
  - We use Newton's laws to set up the equation of motion for x.
  - If all goes well we find that this equation can be written in the form

$$\ddot{x} = -\text{ something-or-other } \times x$$

  where the something-or-other is determined in terms of the properties of the system.

- We conclude that the x-coordinate of this system will exhibit SHM with angular frequency

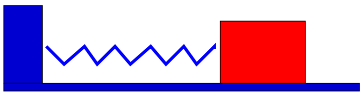
$$\omega = \sqrt{\text{something-or-other}}$$

- This conclues the job: anything else we need is already available in our general SHM results.

- The following applications show this strategy at work.

[B] Application: mass on spring

A mass m rests on a horizontal frictionless surface, and is attached to a wall by a spring of force constant k. Establish the motion if the mass is pulled a small distance (stretching the spring from its natural length), and then released.
[Yes: you have seen this example before; it is useful to help establish the sequence of steps in the strategy; you should work through it yourself first; then check what you have done against the on-line notes .]



← SOLUTION

Results

KEY POINT **6.4**
A mass m subject to a linear spring-force of spring-constant k will exhibit SHM of angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

Check It!

- Are the units OK? ☐
- Does it make sense? ☐

← COMMENTARY

← M

[C] Application: hydrogen molecule

A molecule of hydrogen can be modelled as two particles of mass $m_H = 1.7 \times 10^{-27}\,kg$ linked by a bond viewed as a spring of spring constant $k = 5.2 \times 10^2\, N \cdot m^{-1}$ Establish the motion if the bond is compressed and then released.

← SOLUTION

Results

The molecular bond (spring) exhibits SHM with angular frequency

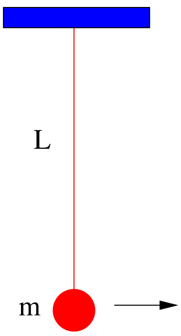
$$\omega = \sqrt{\frac{2k}{m}} = 7.8 \times 10^{14} rad \cdot s^{-1}$$

← COMMENTARY

← I

[D] Application: simple pendulum

A simple pendulum comprises a point mass m suspended from a fixed point by a string of length L. Establish the motion if the bob is pulled to one side and then released.



← SOLUTION

Results

KEY POINT **6.5**
A pendulum comprising a point mass suspended from a fixed point by a light string of length L exhibits SHM of angular frequency

$$\omega = \sqrt{\frac{g}{L}}$$

provided the angular displacement from the vertical is small.

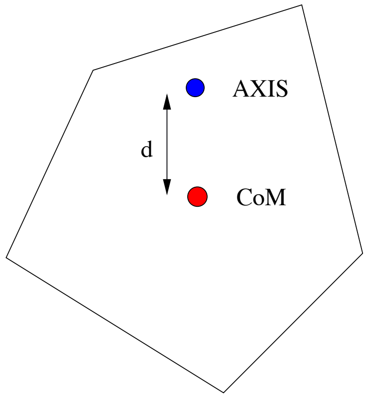
Check It!

- Are the units OK? ☐
- Does it make sense? ☐

← COMMENTARY

← M

[E] Application: physical pendulum



A physical pendulum is made from a sheet of steel, of mass m, pivoted about an axis through a point a distance d from its centre of mass. The moment of inertia about this axis is I. Establish the motion if the sheet is pushed to one side and then released.

←SOLUTION

Results

The physical pendulum exhibits SHM of angular frequency

$$\omega = \sqrt{\frac{mgd}{I}} \tag{6.8}$$

provided the angular displacement from the vertical is small.

←COMMENTARY

←REVIEW

[F] The core physics of SHM: stiffness versus inertia

- We can identify a common underlying structure in these examples.

> KEY POINT **6.6**
> The SHM frequency ω is determined by the properties of the system; it can be written in the generic form
> $$\omega = \sqrt{\frac{\text{measure of stiffness of system}}{\text{measure of inertia of system}}}$$

- By 'stiffness' we mean strength of effects restoring equilibrium
- By 'inertia' we mean strength of effects resisting changes in motion
- Oscillation results from an interplay between the two: 'stiffness' favours return to equilibrium; inertia results in overshoot beyond equilibrium.
- While 'stiffness' and 'inertia' may each take different forms, if correctly identified their ratio has units $1/[\text{second}]^2$
- These insights are enough to allow one to estimate frequencies for 'new' problems...with virtually no mathematics!

← EXAMPLE

← T

Learning Resources

- **Textbook:** HRW 15.5-15.6
- **Self-Test Questions:** available on–line
- **Course Questions:** The familiar mass spring system, A different mass-spring system?, Still more mass-spring systems, Thinking about the pendulum, Swinging stick, Swinging leg, Bouncing a ball.

S6.5 Energy conservation in SHM

[A] Why SHM entails energy conservation

- In section S6.2 we saw that SHM emerges when we have a mechanical system with an associated potential energy:
  - The minimum of the PE identifies an equilibrium position.
  - SHM results if the system is displaced a little from that point, provided the equilibrium is stable.
- If the only force that affects the motion is the one associated with this potential energy then the system is conservative.
- In section S3.7 we saw that the total (kinetic and potential) energy in such a system is conserved (ie constant).
- We must thus expect that, quite generally, mechanical energy is conserved within SHM.
- We show this explicitly for the mass-spring system.

← ANALYSIS

KEY POINT **6.7**
In a mechanical system exhibiting SHM the sum of the kinetic and potential energy remains constant at its initial value:

$$K + U = E = \frac{1}{2}mx_m^2\omega^2 = \frac{1}{2}kx_m^2$$

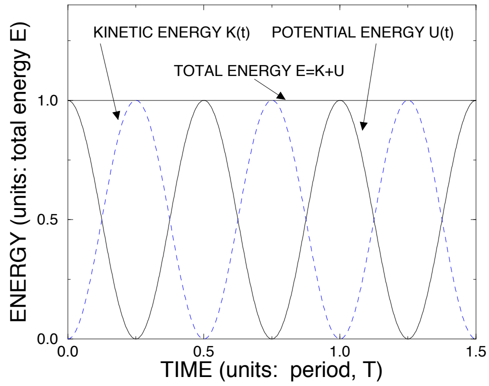
← COMMENTARY

[B] Variation of K and U during the SHM cycle

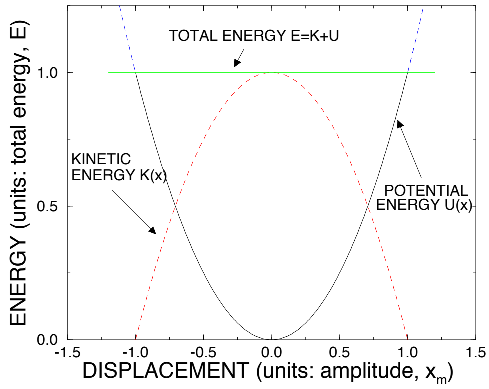
- Although the sum of K and U remains constant during the SHM cycle each of them varies individually.

- One can think of energy being continually exchanged between the two forms.

- We can display the variation as a function of time or as a function of space (displacement).



- The exchange of KE & PE as function of t is shown in the figure
- Each energy has a maximum twice in each cycle.



- The exchange of KE & PE as function of x. is shown in the figure
- One can make this picture less abstract by thinking of a particle sliding on a frictionless parabolic surface.

- The 'distance' between the $x-$axis and the parabola is the potential energy
- The 'distance' between the parabola and the total energy is the kinetic energy.
- The particle never gets beyond $\pm x_m$ because its kinetic energy runs out there.
- This turns out to be a useful platform for thinking about the quantum world ...

$\leftarrow\boxed{\text{I}}$

[C] From energy conservation to the SHM equation

- We have shown that a system exhibiting SHM displays conservation of mechanical energy.

- We can reverse this argument:

ANALYSIS

KEY POINT **6.8**
If the total energy of a system can be written as

$$E = U + K = \frac{1}{2}Sx^2 + \frac{1}{2}I\dot{x}^2 = \text{ constant}$$

where S and I are constants, then the coordinate x will exhibit SHM with frequency

$$\omega = \sqrt{\frac{S}{I}}$$

COMMENTARY

T

Learning Resources

- **Textbook:** HRW 15.4
- **Self-Test Questions:** available on–line
- **Course Questions:** Energy in SHM, Water oscillations in a U-tube, Consolidation exercise.

S6.6 Driving and damping

[A] An overview: the ins and outs of energy

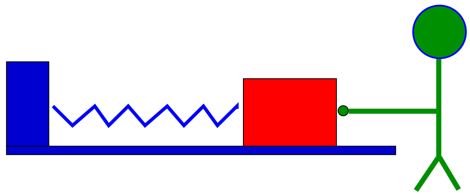
- Thus far we have assumed that our oscillator is effectively isolated from anything else: its energy is thus constant.
- We must now allow for two possibilities:
  - There is some mechanism that feeds energy into the oscillating system: we call this driving.
  - There is some mechanism that draws energy out of the oscillating system: we call this damping.
- We shall initially consider these two mechanisms separately; then together.
- We shall explore, but not prove the main results.

[B] Feeding energy in: driving

- Consider a simple harmonic oscillator of natural frequency ω subjected to a driving force which oscillates at frequency ω_D

$$F_{\text{driving}} = F_D \cos(\omega_D t) \qquad (6.9)$$

- The figure shows one example ...but there are many others.



DRIVING MECHANISM

← EXAMPLES

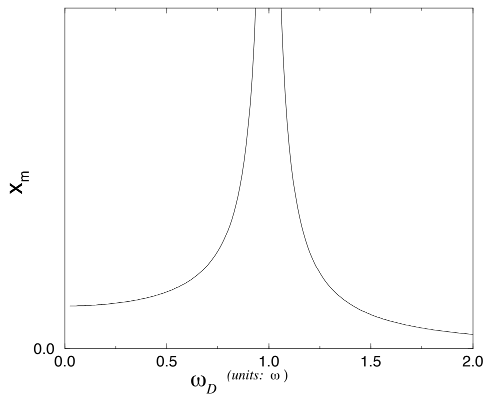
- The equation of motion:

$$m\ddot{x} = -kx + F_D \cos \omega_D t$$

- The resulting behaviour:
  x exhibits SHM but
  - at the frequency ω_D (not ω)
  - with an amplitude that is determined by

$$x_m = \left| \frac{F_D/m}{\omega^2 - \omega_D^2} \right| \qquad (6.10)$$

  - If ω_D is close to ω the amplitude x_m is large. This is called resonance.



KEY POINT **6.9**
If a system of natural oscillation frequency ω is driven by some disturbance oscillating at frequency ω_D the system will oscillate at ω_D with an amplitude that is large if ω_D is close to ω.

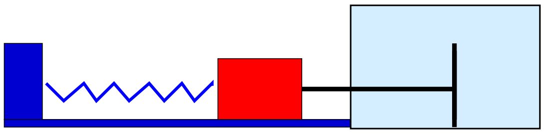
← COMMENTARY

[C] Drawing energy out: damping

- Consider a simple harmonic oscillator of natural frequency ω subjected to a damping force proportional to the instantaneous velocity, and in the opposite direction to it:

$$F_{\text{damping}} = -b\dot{x} \qquad (6.11)$$

where b is a damping constant



← EXAMPLES

- The figure shows one example

  ...but there are many others.

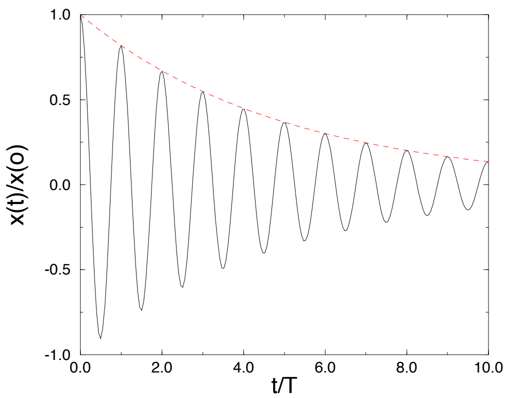
- The equation of motion:

$$m\ddot{x} = -kx - b\dot{x}$$

- The resulting behaviour: x exhibits SHM close to the natural frequency ω but with an amplitude that decays exponentially with time

$$x_m(t) = x_m(0)e^{-\gamma t/2} \qquad (6.12)$$

where $\gamma \equiv b/m$



← COMMENTARY

[D] Energy balance: damping and driving

- Now we consider what happens in a system in which we have both driving (as in Equation 6.9) and damping (as in Equation 6.11.)

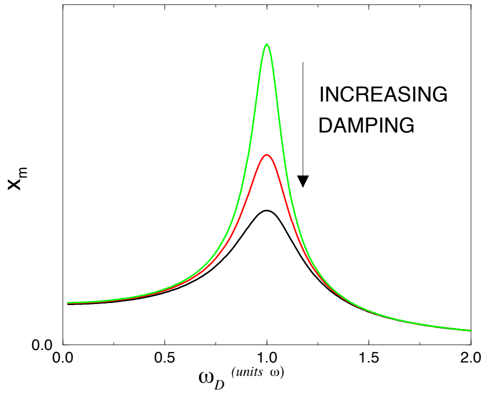
- The equation of motion:

$$m\ddot{x} = -kx + F_D\cos(\omega_D t) - b\dot{x}$$

- The resulting behaviour:

x exhbits SHM

- at the frequency ω_D (not ω)
- with a phase that is different from that of the driving force
- with an amplitude that is large, but not infinite at resonance.



← COMMENTARY

← M

Learning Resources

- **Textbook:** HRW 15.8-15.9
- **Self-Test Questions:** available on–line
- **Course Questions:** Damping, Resonance: the downside.

S6.7 Chaos

This section is included for general interest. It is not part of the examinable programme of the course.