Session 8

Maxwell's Equations

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Contents

Welcome

Welcome to session 8. In this session we're going to look at how electricity and magnetism can be unified into a system of equations, named after James Clerk Maxwell, the Scottish physicist who first proposed them. Then we'll see how this leads to an understanding of the nature of electromagnetic radiation.

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Learning Objectives

- Explain how Maxwell's theory leads to electromagnetic waves
- Show a knowledge of the Poynting vector and electromagnetic energy
- Describe the states of polarisation of an electromagnetic wave
- Describe how reflection and refraction are accounted for by the wave theory
- Give an account of reflection and refraction at a plane boundary including Snell's law

The Problem

The problem involves the search for leaks in water pipes crossing desert areas. Visual human inspection is costly and inefficient. The problem with using satellite or aircraft images is in interpreting the shadows which can be due to many other things than leaking pipes. The idea to be investigated is whether this can be solved by using robotic vehicles to look for changes in refractive index in wet sand. At this stage of the investigation we are not interested in the practicalities of the robotics, only in the theoretical considerations of the interaction of electromagnetic radiation with wet and dry sand.

Image[1]

In order to solve the problem it's clear that we need to know what is meant by a refractive index and how that can affect the behaviour of electromagnetic waves at a surface. To study this we'll have to start by understanding electromagnetic waves and to investigate their properties we'll need a mathematical treatment.

[1] Pipeline, Andrewcparnell, as poster on www.flickr.com. Creative Commons Licenced.

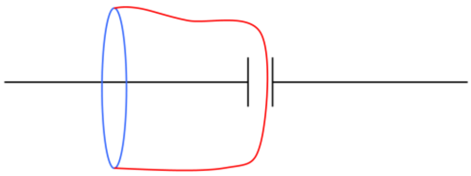
Electromagnetic Waves

Maxwell's Equations

The key insight into the relationship between light and oscillations of the electromagnetic field was obtained by Maxwell. The insight involves returning to Ampere's law in a special circumstance, namely a simple circuit consisting of a long wire through which charge is flowing into a capacitor.

Key is to modify Ampere's law.

Circulation of magnetic field = current



Infinite wire with capacitor

Recall that Ampere's law says that the circulation of the magnetic field is the enclosed current – up to a constant factor that depends on the units. We've drawn two surfaces through which to calculate the current. As the capacitor charges up the current through the blue surface will be non-zero. On the other hand the current through the red surface that threads the capacitor appears to be zero. This would violate Amperes law.

The flux of charge appears to depend on the surface!

$$2\pi r B_\phi = \mu_0 I = \mu_0 \frac{dQ}{dt} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad (1)$$

Circulation of ***B*** d/dt (Flux of ***E)***

To restore the law Maxwell modified it to include the field that flows between the capacitor plates as the charge on the plates builds up. We get the exact expression by using Gauss's theorem to relate the charge on the plates to the flux of the electric field E: Recall that Gauss's theorem says that, up to a constant, the flux of the electric field through a closed surface equals the enclosed charge. So the rate of change of flux will equal the rate of change of charge; that is the current. In equation (1) we've therefore replaced the current in Ampere's law by the rate of change of electric flux. In this form the law will be valid for surfaces such as the red surface.

In order that Ampere's law remains valid under all circumstances we allow both the real currents and the electric flux to contribute to the magnetic circulation. Ampere's law becomes: The Circulation of the magnetic field B = the Current + the Rate of change of the Flux of the electric field E. Because the rate of change of the flux of the electric field acts as a current in Ampere's law, it is called the displacement current.

So Ampere's law becomes

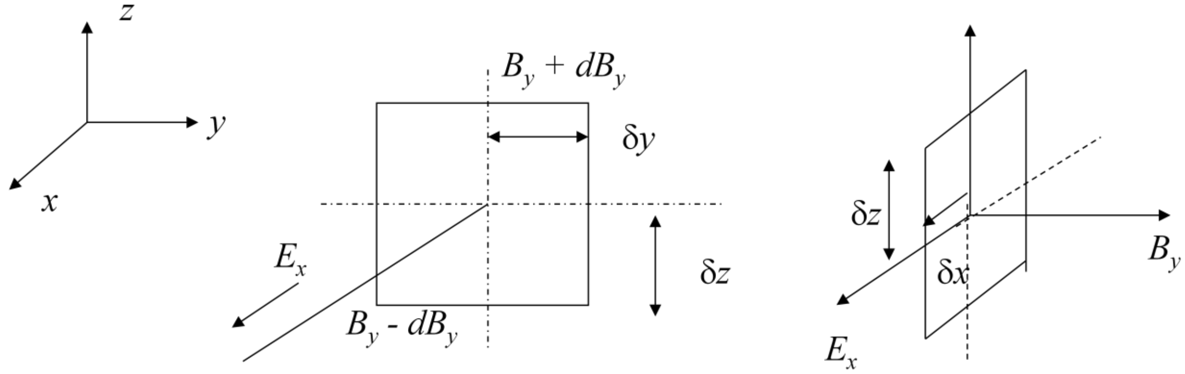
Circulation of ***B*** = Current + Rate of change of Flux of ***E***

("displacement current")

Derivation of Maxwell's Equations

To get Maxwell's equations we'll look at a special case. In this special situation we have an electric field with only an x-component E_x – the y and z components of E are set to be zero. In addition we'll assume that E_x doesn't depend on where in the xy plane it is measured: we'll assume it's always the same. In other words E_x doesn't depend on x or y. But we do let it depend on the z coordinate. We also let it change in time. So in this situation E_x is assumed to be a function of z and t. We'll see why this is a good thing to do in a moment. How should we simplify the magnetic B field? By trial and error we find that a useful assumption is that B has only a y component, B_y, and that this also depends only on z and the time, t.

- apply Faraday's law and Ampere's law to small squares



We're now ready to write down Faraday's law for a small square in the yz plane and Ampere's law for a small square in the xz plane. Why do we choose these planes? Because for Ampere's law we want the flux of the E field so we choose a surface perpendicular to E_x and for Faraday's law of induction we want the flux of the B field, so we choose a surface perpendicular to B_y.

We're now in a position to derive Maxwell's equations for this case.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

Application to Electromagnetic Fields

Where have we got to? – We have a system of equations for the electric and magnetic fields in a particular situation.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad (1)$$
$$\frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \qquad (2)$$

But what does all this mean?

Look at the equations. Equation (1) says that a time varying B field, of the right form, generates an E field. This is just Faraday's law of induction. Equation (2) says that the E field regenerates the B field. Why? Because the E-field is acting as a current in Ampere's law.

This is still not a particularly intuitive picture. So we'll take the maths a bit further. We're now going to derive the most important consequence of Maxwell's equations, from which we'll see that what we've been talking about are in fact electromagnetic waves.

We can derive the equation governing wave motion that we met in session 3 by manipulating equations (1) and (2). We begin by differentiating (2) with respect to t and then substitute for dB/dt from equation (1).

$$\frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E_x}{\partial t^2} \underset{\text{from (2)}}{=} -\frac{\partial}{\partial t} \frac{\partial B_y}{\partial z} \underset{\text{Interchange derivatives}}{=} -\frac{\partial}{\partial z} \frac{\partial B_y}{\partial t} \underset{\text{from (1)}}{=} \frac{\partial}{\partial z} \frac{\partial E_x}{\partial z} = \frac{\partial^2 E_x}{\partial z^2}$$

You may recognise the resulting equation (3) – apart from the use of E_x instead of y as the dependent variable, and the use of z instead of x as the coordinate describing the spatial dependence, this is, what we called in section 3, the wave equation.

$$\text{Wave equation for } E_x\text{:} \quad \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad (3)$$

The speed of propagation of the wave is $1/\sqrt{\mu_0 \varepsilon_0}$ which is numerically very close to the speed of light. Since there is nothing that has a speed close to that of light that isn't in fact light, we take it that these equations describe light in terms of time-varying electric and magnetic fields.

Work out $(\mu_0\ \varepsilon_0)^{-1/2}$

Starting from (1) derive the wave equation for B_y

Checking solutions

As a check we can show that this equation has a wave-like solution. We let the electric field E_x have the usual form of a harmonic wave in equation (1) and compute the second derivatives with respect to z and t as shown.

Let $\quad E_x = E_0 \cos(\omega t - kz) \quad$ where E_0 is a constant $\quad$ (1)

$$\frac{\partial E_x}{\partial t} = -E_0 \omega \sin(\omega t - kz) \qquad \frac{\partial^2 E_x}{\partial t^2} = -E_0 \omega^2 \cos(\omega t - kz)$$

$$\frac{\partial E_x}{\partial z} = E_0 k \sin(\omega t - kz) \qquad \frac{\partial^2 E_x}{\partial z^2} = -E_0 k^2 \cos(\omega t - kz)$$

The wave equation specifies how the second derivatives are related. We can check that the required relation in equation (3) of the previous section will be satisfied if ω and k are related as in equation (2) where we've put c for the speed of light.

So E_x satisfies the wave equation provided that

$$\omega^2 = c^2 k^2 \qquad (2)$$

where $c^2 = 1/(\varepsilon_0\ \mu_0)$

The wave equation for the magnetic field B will be satisfied by a similar harmonic wave. But what is the relation between the phases of B and E? Let's see.

Relation between E and B

To get the relation between E and B we go back to Maxwell's equations.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

We'll guess the relation and then check it. We've guess that if E is proportional to cos(ωt-kz) then so is B. We've checked the first of Maxwell's equations below and found that the amplitude of the magnetic oscillation is 1/c times smaller than the electric component. This is just what we might expect from our investigation of the Biot Savart law is session 5. We'll leave you to check the second Maxwell equation.

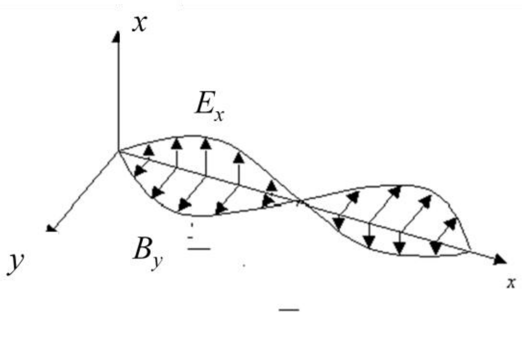
Let $B_y = B_0 \cos(\omega t - kz)$ where B_0 is a constant (1)

Then $\frac{\partial B_y}{\partial t} = -B_0 \omega \sin(\omega t - kz) = -\frac{\partial E_x}{\partial z} = -E_0 k \sin(\omega t - kz)$

if $B_0 = E_0/c$

i.e. if $B = E/c$ or $\mathbf{B} = \boldsymbol{k} \times \mathbf{E}/c$

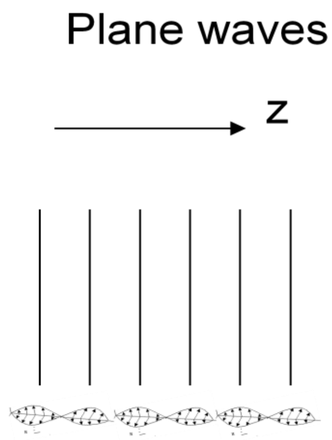
where $\boldsymbol{k}$ is a unit vector in the direction of propagation



Our second conclusion from this calculation is that E and B are IN PHASE. A propagating electromagnetic wave therefore looks like the illustration in the figure, with E and B oscillating together.

The Electromagnetic Spectrum

The waves we are talking about here are called plane waves because the fields are constant on xy planes. We can think of the xy planes as forming the wavefronts progressing in the z direction as shown here:

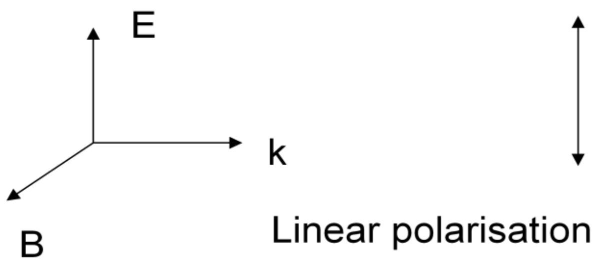


The parameters k and ω can take any values so long as $\omega/k = c$, the speed of light. So the wavelength of the waves can range from as large as you please to as small as you please. Thus, Maxwell's equations describe the whole electromagnetic spectrum from radio waves to gamma rays: they are all oscillations of the electric and magnetic fields.

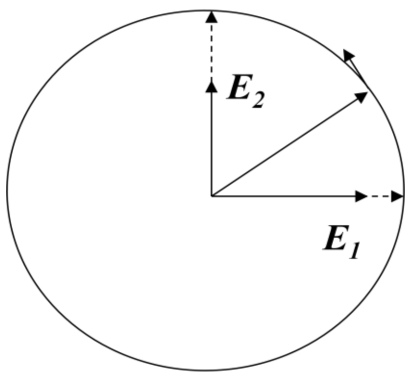
Polarisation

Electromagnetic waves come in more complex forms than the simple infinite plane waves that we have described so far. We're not going to take that any further here. But there is one complication we'll need – the polarisation of the wave.

So far we've considered an E field in the x direction and a B field in the y direction. Suppose we swap the two to get an E field in the y direction and a B field in the x direction. This is obviously a solution of Maxwell's equations because we can assign the x and y labels for the axes as we like. We describe the difference between the two waves by saying that they exhibit different states of linear polarisation. The convention is that the direction of the E field is used to describe the polarisation state – hence that E field (and the polarisation) can be vertical or horizontal.



Recall that the E field isn't static – this is a wave, so at any point the E field is oscillating in amplitude with time. We can add two waves with the E fields at right angles, and with a 90^0 phase difference, to get a polarisation vector that rotates. In this case we say that the wave is left or right circularly polarised.



(Left) Circular polarisation – the E vector rotates anticlockwise looked at along the direction of motion

You've probably some experience of polarisation – for example, sun glasses are polaroids. So you know that the interaction of light with a material can depend on its polarisation. This could be helpful to us in the problem of identifying wet sand, but we first need to understand the interaction of radiation with materials. We'll take that up in the next section.

Show how two orthogonal E vectors can be added to produce a right circularly polarised wave.

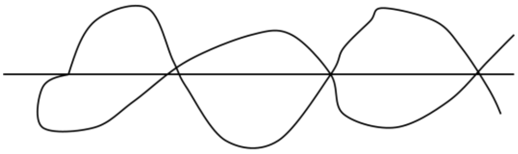
Summary

- Ampere's law must be modified to take account of the "displacement current" that arises from a time-varying electric field, in addition to real currents.
- When this is done, it is possible for time varying electric and magnetic fields to generate each other in a vacuum
- This gives rise to electromagnetic waves which we identify as moving with the speed of light
- Therefore light waves are oscillating electric and magnetic fields
- The direction of the E-vector defines the state of polarisation of the wave

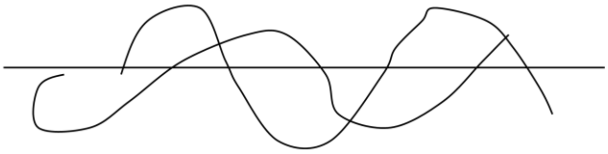
SAQs

1. If $E_x = E_0 \sin(\omega t - kx)$, B_y =
   a) $(E_0/c) \sin(\omega t - kx)$
   b) $(E_0/c) \cos(\omega t - kx)$
   c) $-(E_0/c) \sin(\omega t - kx)$
   d) d) $-(E_0/c) \cos(\omega t - kx)$

2. Which of the following is the correct picture of a propagating electromagnetic wave?



(a)



(b)

The answers appear on the following page

Answers

1. (a) correct: B and E are in phase so if E is a sine wave, so is B.

   (b) incorrect: If E is a cosine wave as in the text, then so is B, but here E is a sine wave, so B must be a sine wave too to be in phase.

   (c) incorrect: the minus sign corresponds to a 90^0 phase difference between E and B which is not present.

   (d) incorrect: the minus sign and the cosine wave would put E and B out of phase

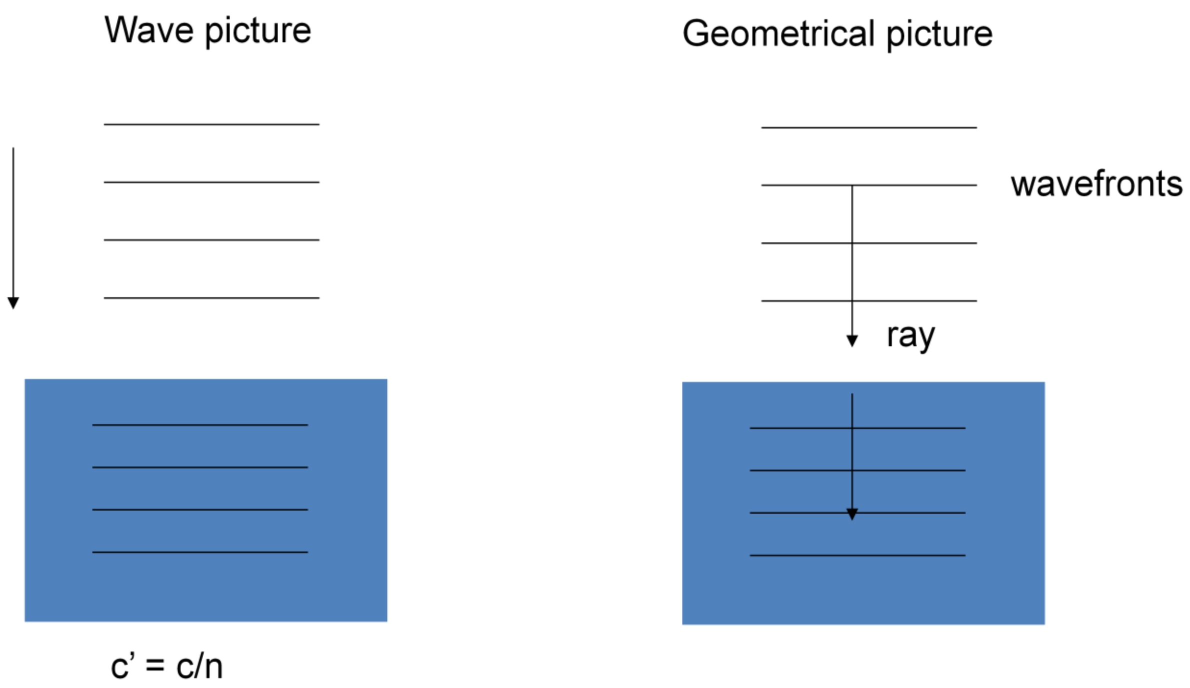
2. (a) correct: This is the same question as 1, but in pictures.

   (b) Incorrect: This is the same question as 1, but in pictures: E and B are in phase.

Fields at Boundaries

Waves in Media

The wave picture on the left shows plane waves incident normally from a vacuum on to a transparent medium. The frequency of the wave is unchanged as it propagates from one medium to the other. The wavelength does change – it is smaller in the medium with higher refractive index. On the right, we have drawn in the normals to the wavefronts. This gives us the ray picture of geometrical optics. We can use the ray representation in situations where there are no wavelike effects - that is, where the light is travelling in straight lines.



The speed of the light is lower in a medium than in a vacuum. This commonly raises a number of issues.

First does the light change colour from one medium to the next? If not, why not? We'll leave you to discuss this.

Second, in the wave picture the light slows down instantaneously on entering the medium – in the photon picture do photons undergo infinite deceleration at the boundary? In fact, they do not. The photons entering the medium interact with it and lose their identity. The particles propagating in the medium are different photons – we'll see this in more detail on the next section.

Dielectric constant ε and refractive index *n*

What causes the slowing down of light in a dense medium? We can represent the medium as a collection of atomic dipoles which oscillate in the presence of an electric or magnetic field. This oscillation contributes to the electromagnetic field in the medium, altering it from the incident field. Consider an oscillating electric dipole. The movement of charge constitutes a current in the medium. This current is only present because of the original oscillating electric field. So, for small fields at least, we can assume that the current depends linearly on the field. We can also assume that it depends on the time derivative of the field, because the dipoles won't oscillate if the field is not varying. Thus, in a medium, we must modify Maxwell's equations by the extra term in equation (2) here. In this equation κ is a constant relating the induced currents to the field. If you work through the maths you'll see that a wavelike solution for E_x now requires that equation (3) is fulfilled. The constant quantity ε is the dielectric constant which we met previously in session 4.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad (1)$$

$$\frac{\partial B_y}{\partial z} = -\frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial E_x}{\partial t} + \kappa \frac{\partial E_x}{\partial t}\right) \qquad (2)$$

$$E_x = E_0 \cos(\omega t - kz) \qquad B_y = B_0 \cos(\omega t - kz)$$

Dispersion relation: $$c^2 k^2 = \omega^2 (1+\kappa) = \omega^2 \varepsilon \qquad (3)$$

Refractive index: $n = ck/\omega = \sqrt{\varepsilon}$ Light speed = ω/k = c/n

The refractive index of a medium, n is defined by n = ck/ω. It is equal to the square root of the dielectric constant. The refractive index is also the speed of light in a vacuum, c, divided by the speed of light in the medium, ω/k. Equivalently, the speed of light in the medium, ω/k, equals c/n. Since refractive indices are greater than unity, this explains why light moves more slowly in a medium.

Why would you expect the refractive index to depend on frequency? (Hint: think back to session 3)

Think about why you might expect the refractive index of a medium to be a function of the frequency of the incident light. ….

The reason can be found in the work of session 3: the way an oscillator responds to an oscillating external force depends on the frequency of the oscillation. So the currents induced in the dielectric by the incident field will depend on its frequency. This means that κ and hence n must be frequency dependent.

Reflection and transmission – normal incidence

We can now look at a wave incident on a dielectric, non-conducting, medium. We'll consider the magnetic field component – the electric field would give identical results. Using our experience that an incident wave will be partly reflected and partly transmitted, we now have three waves to consider: the incident wave, the reflected wave and the transmitted wave.

$$B_{\mathrm{inc}} = B_i \cos(\omega t - k_i z); \quad B_{\mathrm{ref}} = B_r \cos(\omega t + k_r z); \quad B_{\mathrm{trans}} = B_t \cos(\omega t - k_t z)$$

We've used the subscripts i, r and t for the amplitudes of these waves and for their wave vectors k. Note that the incident and transmitted waves are travelling in the positive z direction, but the reflected wave is travelling in the opposite direction : hence the argument ωt + kz of the cosine function, with the opposite sign for k. We'll assume for simplicity that the incident medium is air with a refractive index of 1.

Now, we have to satisfy the wave equation on the boundary, because the laws of electromagnetism are supposed to hold everywhere. So we'll need to find the second derivative of B. That means that on whichever side of the boundary we calculate it we must get the same answer. Thus, dB/dz must be the same on both sides of the boundary, and B itself must be too. On the incident side the total B field is made up of the sum of the incident and reflected wave. So the derivative of this sum must equal the derivative of the field on the transmission side on the boundary at z=0. This leads to equation (1). Similarly the total field must be the same on both sides the boundary at z=0. This leads to equation (2).

$\frac{\partial B}{\partial z}$ must be continuous (to form the wave equation) so

$$\frac{\partial}{\partial z}[B_i \cos(\omega t - k_i z) + B_r \cos(\omega t + k_r z)] = \frac{\partial}{\partial z}[B_t \cos(\omega t - k_t z)] \quad \text{at } z = 0$$

$$k_i B_i - k_r B_r = k_t B_t \quad \text{or} \quad B_i - B_r = B_t / n \qquad (1)$$

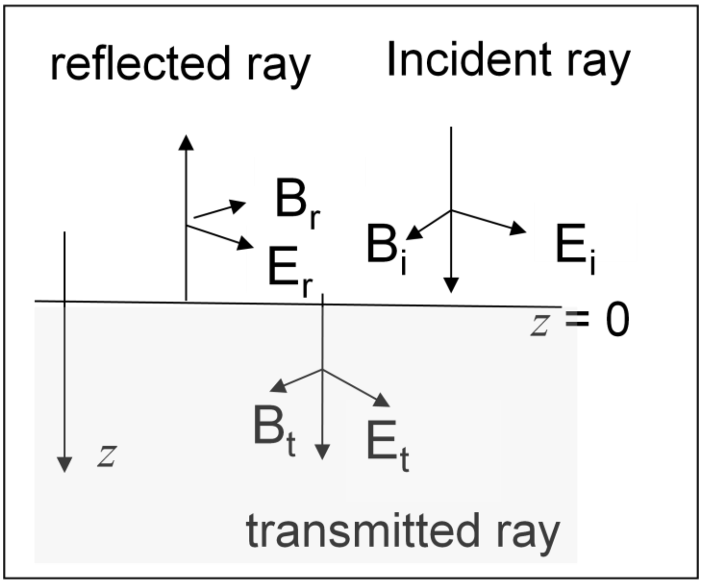
B must be continuous to form $\frac{\partial B}{\partial z}$ so $\quad B_i + B_r = B_t \qquad (2)$

Equations (1) and (2) can be solved for the reflected field and the transmitted field in terms of the incident field. The ratio of the field transmitted to that reflected depends on the refractive index.

Solving gives

$$B_t = \frac{2n}{n+1} B_i \quad \text{and} \quad B_r = \frac{n-1}{n+1} B_i$$

However, we don't normally measure radiation fields directly – we usually measure the energy flow. So how do we convert fields to energy fluxes?



Energy Carried by a wave

Just as in session 3, the energy of a wave depends on the square of the amplitude. For the electric and magnetic fields we've met the expressions for the energy densities in previous sessions. The energy per unit volume is given by equation (1). To get the flux of energy we multiply this by the speed of the wave. We saw in the section relating the dielectric constant to the refractive index that this is c/n in a medium of refractive index n. So we're now in a position to give the reflected and transmitted energies when a beam of radiation strikes a dielectric.

In a vacuum the energy is $\mathscr{E}$

$$\mathscr{E} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\frac{B^2}{\mu_0} \qquad (1)$$

The energy depends on the square of the field

The flux of energy across a surface is $(c/n)\,\mathscr{E}$ (2)

This gives the transmission coefficient

$$T = \frac{\text{transmitted flux of energy}}{\text{incident flux of energy}} \qquad T = \frac{4n}{(n+1)^2}$$

and reflection coefficient

$$R = \frac{\text{reflected flux of energy}}{\text{incident flux of energy}} \qquad R = \frac{(n-1)^2}{(n+1)^2}$$

Show that *T* + *R* =1. What does this equation mean physically?

How far do waves penetrate a conductor?

So far we have considered waves incident on perfectly insulating media, that is we have neglected the effects of conductivity. In a real medium any incident radiation will not only shake the atomic dipoles but will also move free charges around. The motion of free charge also constitutes a current that will add to the field. We know that a static field inside a conductor is zero, because the charges move to neutralise any field. If the charges respond to a time varying field entirely without resistance then they will instantaneously neutralise a time varying field inside the conductor as well. However, if the conductivity is finite, conduction currents will flow under the influence of the electric vector of the wave. This current will dissipate energy through ohmic heating, so the wave will lose energy as it propagates. The distance over which the amplitude of the E vector falls to 1/e of its initial value is called the skin depth. The skin depth is given by equation (1).

Skin depth (for low frequency waves)

$$\delta = \sqrt{\frac{2\varepsilon_0 c^2}{\sigma\omega}} \qquad (1)$$

Why is this relevant to our problem of detecting leaking pipes? – From this point of view, water is a conductor. Thus if em radiation penetrates wet sand it may be reflected not from the sand surface, but from buried objects – such as the water pipe itself!

At low frequencies, for example for radio waves, the skin depth of a wave of frequency ω in a medium of conductivity σ is given by equation (1). In a good conductor the skin depth is quite small. In copper for example the skin depth is about 1/100 th of a mm.

Example: a copper conductor

$\sigma = 5.76 \times 10^{7}$ ohm^{-1} metre^{-1}

$\varepsilon_0 = 8.85 \times 10^{-12}$ farad metre^{-1}

$c = 3 \times 10^{8}$ m s^{-1}

$\omega = 2\pi \times 10^{10}$ rad s^{-1} (microwaves)

Then: $\delta = 7 \times 10^{-6}$ m

Summary

- Waves in a medium with refractive index n travel at speed c/n
- The refractive index is the square root of the dielectric constant and depends on frequency
- The energy in the electromagnetic field is: $\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$
- The ratio of transmitted energy to reflected energy depends on n
- The energy travels at speed c/n in a medium
- Waves penetrate into a conductor a distance given by the skin depth: $\delta = \sqrt{\frac{2\varepsilon_0 c^2}{\sigma\omega}}$

SAQs

1. Why is it difficult to send high frequency electromagnetic waves down copper wires?

   a) The electrons move too slowly

   b) The resonance width is too low

   c) High dissipation leads to confinement of the field

   d) The signal is totally internally reflected

2. Why is the skin depth for water different from a copper conductor?

   a) Water is a poor conductor

   b) Water is transparent

   c) Because the dielectric constant of water is larger

3. Communication with submarines from a base station on land (using the electromagnetic spectrum) requires

   a) High frequency radiation

   b) Low frequency radiation

   c) High power at any frequency

4. For normal incidence the polarisation of the reflected wave is:

   a) the same as that of the incident wave

   b) 90^0 to the incident wave

   c) random (i.e always unpolarised)

The answers appear on the following page

Answers

1. a)Incorrect: the signal is not carried by translational motion of electrons, but by oscillations communicated at speed c/n

   b) Incorrect: the conduction electrons in a metal are free so the resonance frequency and width are irrelevant

   c) correct: the conductivity of copper is high, so the electric vector gives rise to large currents and hence large I^2R losses. Fields do not therefore penetrate very far so the skin depth in a metal is very small. Thus electromagnetic waves are confined to the surface layer.

   d) incorrect: it would be relevant if one were to try to shine a beam down the copper wire, because copper like all metals is a good reflector as a consequence of the small skin depth. However, the signal would be generated electromagnetically in the wire.

2. a) correct: the formula for skin depth involves the conductivity in the denominator so high conductivity implies small skin depth. This is also physically reasonable because the high conductivity implies large currents, hence large I^2R losses and small penetration.

   b) incorrect: water is transparent to optical wavelengths because the skin depth at these wavelengths is large, not the other way round.

   c) the statement is true but not relevant because the dielectric constant doesn't affect the skin depth.

3. a) incorrect: this is another skin depth problem since the em waves have to penetrate the ocean.

   b) correct: only the lowest frequencies can penetrate to depth in sea water (which has a relatively high conductivity compared to tap water, say. Pure water is non-conducting.)

   c) incorrect: the signal fall off exponentially with distance so while high power might have some effect it's not sufficient on its own

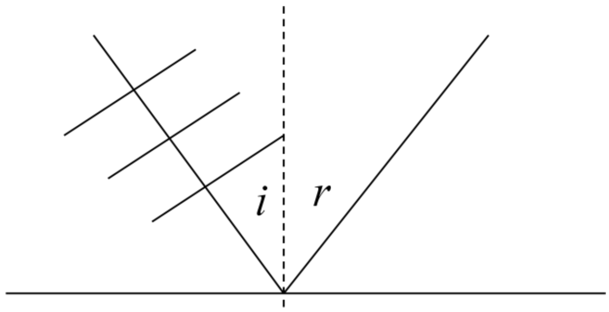
4. a) correct: the E and B fields each have to be continuous across the boundary which means that they each have to align independently

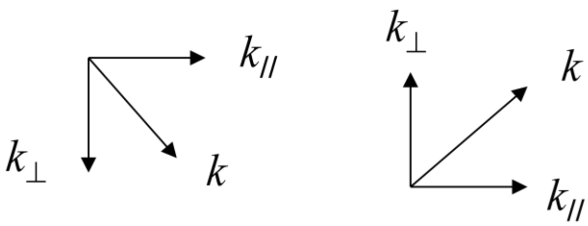
   b) and c) incorrect: a wave may be polarised on reflection by selective polarisation-dependent reflectivity but the polarisation of an individual wave doesn't change on reflection.

Reflection and Refraction

Oblique Incidence

The first observation we usually encounter in discussing reflection is the equality of the angles of incidence and reflection. Why are these equal?



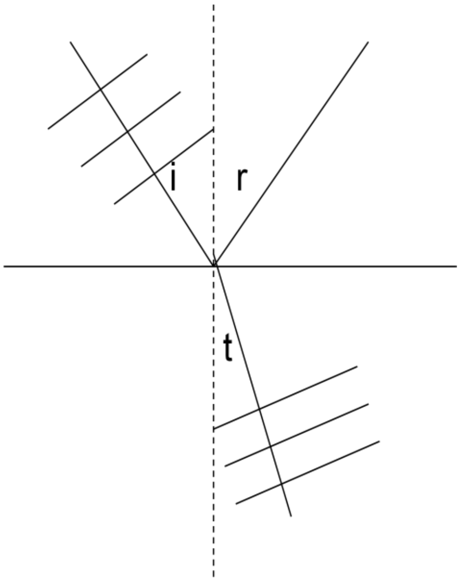


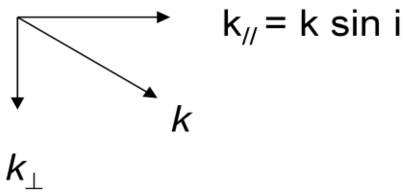
We can think of the wave vector k which gives the direction and wavenumber (or wavelength) of the wave as having components parallel to the surface and perpendicular to it. Let's take the example of a perfect conductor. The reflected and transmitted waves have to cancel along the surface, so $k_{//}$ must be the same for the incident and reflected waves. But the wave numbers of the reflected and incident waves are also the same in the same medium. So $k_{\perp}$ must be the same for both. This means the angles are the same.

Where does Snell's Law come from?

Turning now to refraction, where does Snell's law come from?

$n_1 \sin i = n_t \sin t$ *(Snell's Law)*





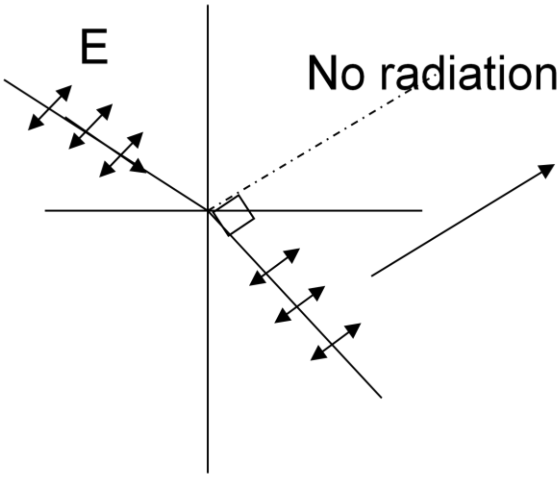
$$k/n = \omega/c = \text{const}$$

$$\text{i.e } k/(n\sin\theta) = \text{constant where } \theta = i, r \text{ or } t$$

Once again, $k_{//}$ must be the same for all waves in order that they cancel along the surface. But k/n is the same in both media, because this is how n is defined. Now, k/n is $k_{//}/(n\sin\theta)$ where θ is the angle of incidence, reflection or refraction. Thus we get Snells law: $n_i \sin i = n_t \sin t$.

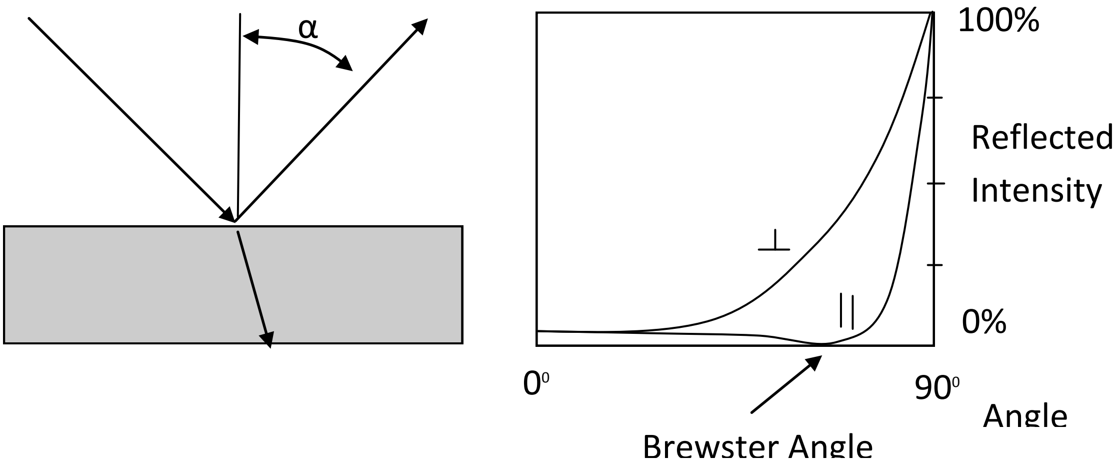
The Brewster Angle

An interesting effect occurs for an incident beam polarised such that the electric field is in the plane of incidence – that is the plane containing the incident ray and the normal to the surface, or the plane of the page as it is usually drawn. The effect is this: at a particular angle of incidence there is no reflected ray.



Brewster angle: $\theta_I = \tan^{-1}(n_t / n_i)$

This can be understood by recalling that reflection occurs because the dipoles in the medium are made to oscillate by the incoming field. The oscillating dipoles produce their own radiation field which contributes the whole of the reflected field and part of the transmitted field. But a dipole does not radiate along its axis. Thus if the transmitted ray is at 90^0 to where the reflected ray would have been, there will be no reflected ray. This occurs at an angle of incidence called the Brewster angle which depends on the refractive indices of the two media.



Poynting vector

Finally we come to another way of looking at the flux of energy which gives us both the direction and magnitude of the energy flow in a field. This is called the Poynting vector. This section shows how it is derived for a plane wave. We start from the usual expression for the energy per unit volume of an electromagnetic field in equation (1). We take the time derivative of the energy in equation (2). We then use Maxwell's equations for a plane wave to eliminate the time derivatives to get equation (3) which is rewritten as (4). This gives us the difference in energy entering and leaving a planar region along the z-axis in terms of the rate of change of energy in the region.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

Energy = $$\mathscr{E} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\frac{B^2}{\mu_0} \qquad (1)$$

$$\frac{d\mathscr{E}}{dt} = \varepsilon_0 E_x \dot{E}_x + \frac{1}{\mu_0} B_y \dot{B}_y \qquad (2)$$

$$= -\varepsilon_0 E_x c^2 \frac{\partial B_y}{\partial z} - \frac{1}{\mu_0} B_y \frac{\partial E_x}{\partial z} \qquad (3)$$

$$= -\frac{\partial}{\partial z}\left(E_x \frac{B_y}{\mu_0} \right) \qquad (4)$$

In general flux of energy is $$\boldsymbol{S} = \boldsymbol{E} \wedge \frac{\boldsymbol{B}}{\mu_0} \qquad (5)$$

$\longrightarrow$ S(z) $\boxed{\dot{\mathscr{E}}}$ $\longrightarrow$ S(z+dz)

Poynting vector

In equation (5) we have given a general expression for the Poynting vector, S, that applies to any situation, not just plane waves.

Summary

For oblique incidence:

- The angle of reflection equals the angle of incidence
- The angle of refraction t obeys Snell's law: $n_i \sin i = n_t \sin t$
- For the electric vector polarised in the plane of incidence there is no reflected ray at the Brewster angle $\tan^{-1}(n_i/n_t)$
- The Poynting vector $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$ gives the energy flux in a wave.

SAQs

1. Unpolarised light is reflected obliquely from a plane surface. What is the state of polarisation of the reflected beam?
   a) unpolarised
   b) E vector parallel to the plane
   c) partly polarized

2. In passing from glass to air a light ray bends
   a) away from the normal
   b) towards the normal
   c) Neither: it is totally reflected

The answers appear on the following page

Answers

1. a) incorrect: the different components of polarisation are reflected more or less strongly so there will be a net polarisation in the reflected beam even if the incident beam is unpolarised.

   b) incorrect: this would be true at the Brewster angle and is a way of producing linearly polarised light.

   c) correct: the different components of polarisation are reflected more or less strongly so there will be a net polarisation in the reflected beam even if the incident beam is unpolarised.

2. a) correct: the easiest way to see this is to trace a ray backwards. The paths are time-reversible because there is no dissipation in geometrical optics (i.e for non-conducting dielectrics or perfectly conducting mirrors).

   b) incorrect: the wave is travelling faster in air than in glass, so the bending is away from the normal.

   c) incorrect: at a large enough angle of incidence this will be true, but it is not the case for small angles of incidence.

Water Pipes in the Desert

What method might we use to determine the difference in refractive index between wet and dry sand? :

Reflection coefficient

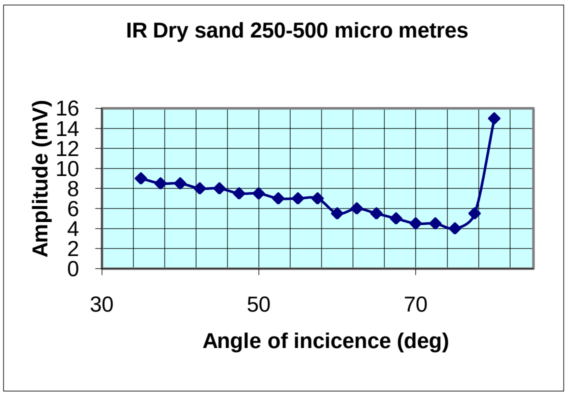
Brewster angle

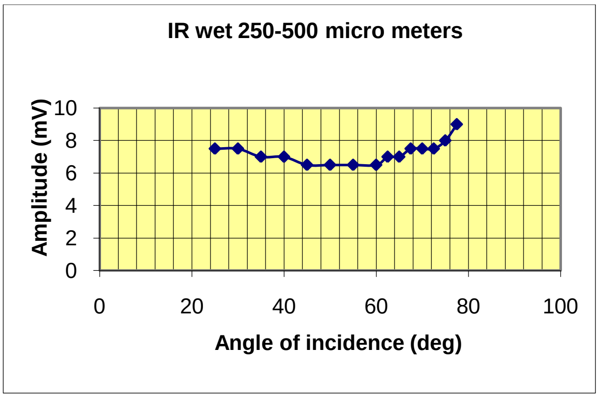
Look up the dielectric constant of water and predict the difference in Brewster angle between wet and dry sand. Why might microwaves be better than optical?

Sand has a refractive index of around 2 to 3. If you look up the dielectric constant of water you'll find a range of values because it is highly dependent on the frequency of radiation. Typical values for the dielectric constant in the microwave region are around 80, whereas the refractive index in the optical we know to be around 1.3, which is far from the square root of 80. This suggests that we might get a larger difference between wet and dry sand if we use microwaves in our detection system.

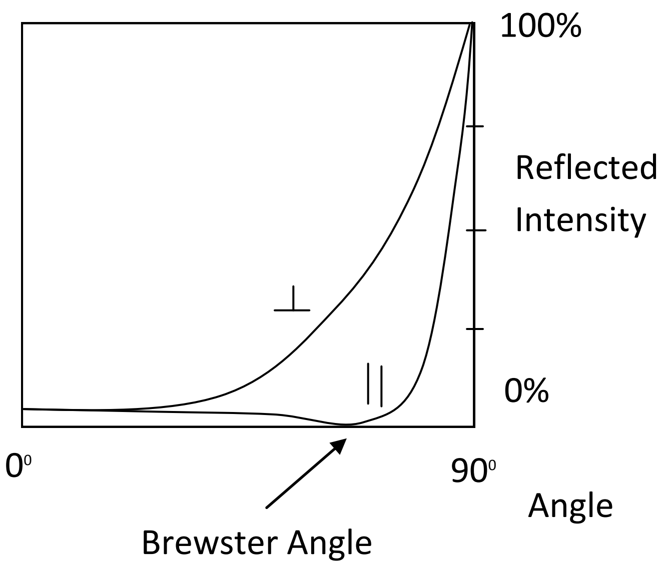
Results for Sand

So here we have some results for the reflection of polarised microwaves from sand. The first thing you will notice is that the curve does not look much like the theoretical curve where we discussed the Brewster angle. Before we attempt to use the method therefore we should try to understand where the discrepancies arise. They are not just experimental error (although the experiment is quite difficult to do accurately).





Theoretical Curve:



$$r_{\perp} = \frac{\sin^2(\alpha - \beta)}{\sin^2(\alpha + \beta)}$$

$$r_{\parallel} = \frac{\tan^2(\alpha - \beta)}{\tan^2(\alpha + \beta)}$$

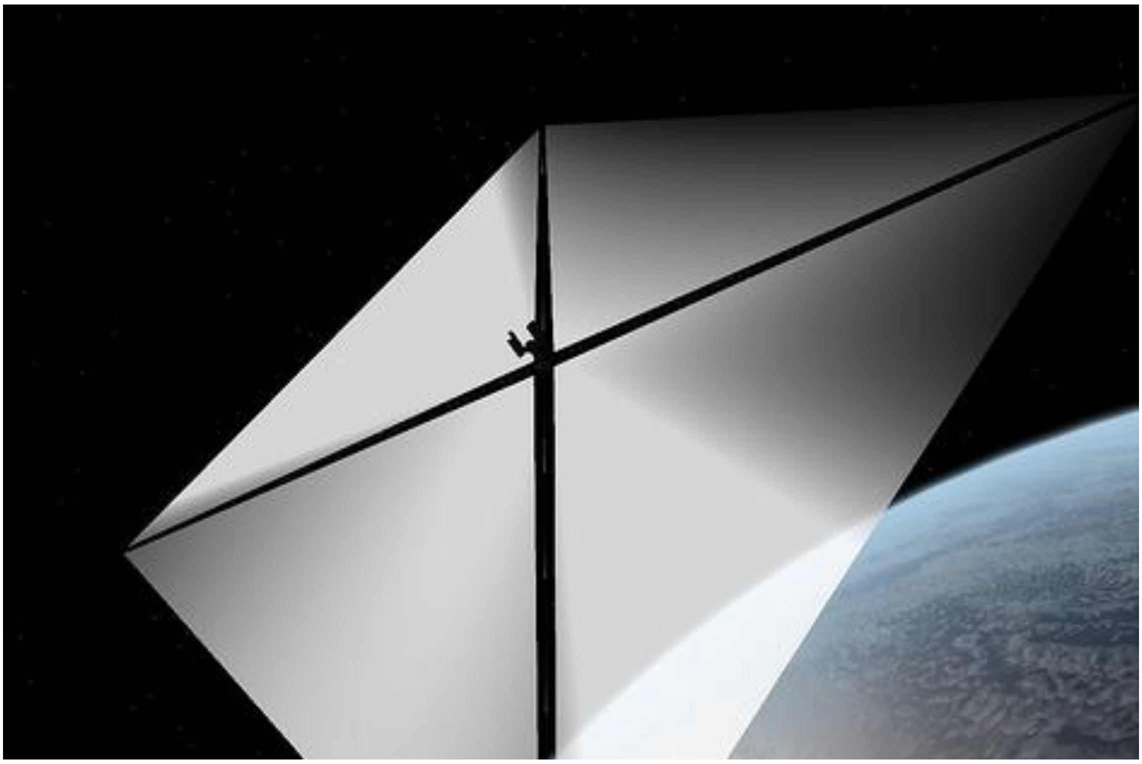
We suggest that you come up with your own ideas on this.

Additional Problems

Problem 1: Solar sails

A solar sail is a means of powering a spacecraft that works by capturing the momentum of radiation from the Sun.

Momentum and energy: the flux of momentum P carried by a beam of light is related to its energy flux E by P=E/c.



Image[2]

Estimate the acceleration of a solar sail at the Earth and comment on your result. (Your answer will depend on the size you assume for your sail.)

Flux of momentum from Sun = $L/(4\pi r^2 c)$;

Force = rate of change of momentum= 2x Area x $L/(4\pi r^2 c)$

Acceleration = Force /Mass

[2] Solar Sail in Orbiter, FlyingSinger, as posted on www.flickr.com. Creative Commons Licenced

Answer: We make the acceleration about 5×10^{-8} m s^{-2} for a sail with area 1000m ^{2} and a spacecraft of mass 100 tonnes. Don't forget the factor 2 for the momentum change on reflection. You might like to think about where the energy comes from that ends up as kinetic energy of the spacecraft. Any practical accelerations are likely to be small, but the method is interesting because it can be used to accelerate a spacecraft over long periods thereby building up significant speeds.

Overall Summary

- Maxwell's equations arise from a modification of Ampere's law to include the displacement current
- Maxwell's equations predict the existence of electromagnetic waves travelling with the speed of light
- The waves are transverse with the electric and magnetic fields orthogonal and perpendicular to the direction of motion
- The speed of an electromagnetic wave in a medium of refractive index n is c/n.
- Maxwell's equations can be used to derive the laws of reflection and of refraction
- At the Brewster angle a wave polarised parallel to the plane of incidence is not reflected
- The Poynting vector gives the energy flux in a wave

Meta tags

Author: Emma Bunce.

Owner: University of Leicester

Title: Enhancing Physics Knowledge for Teaching – Maxwell's Equations

Keywords: Magnetism; Electricity; Magnetism; Fields; sfsoer; ukoer

Description: In this session we're going to look at how electricity and magnetism can be unified into a system of equations, named after James Clerk Maxwell, the Scottish physicist who first proposed them. Then we'll see how this leads to an understanding of the nature of electromagnetic radiation.

Language: English

Version: 1.0

Additional Information

This pack is the Version 1.0 release of the module. Additional information can be obtained by contacting the Centre for Interdisciplinary Science at the University of Leicester.
http://www.le.ac.uk/iscience